Pearce's Lemma
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Agenda

- Introduction
- Definitions
- Proof
- Appendix: Weierstrass' extreme value theorem

A Characterization of Rationality

Pearce's Lemma:
The rational choices in a static game are exactly those choices that are not strictly dominated.

Application

Four ways to rationality:

1. Identify all rational choices: find a belief on the opponents' choices such that the respective choice is optimal.
2. Identify all irrational choices: show that the respective choice is not optimal for any belief on the opponents' choices.
3. Identify all choices that are not strictly dominated: find an opponents' choice-combination such that there is no choice that is better than the respective choice.
4. Identify all choices that are strictly dominated: show that the respective choice fares worse than some other choice for all opponents' choice-combinations.

Note:

- For rational choices it is often easier to find a supporting belief.
- For irrational choices it is often easier to show strict dominance.

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Games

Definition

A *static game* is a tuple

$$\Gamma = \big(I, (C_i)_{i \in I}, (U_i)_{i \in I}\big),$$

where

- I denotes the finite set of *players*,
- C_i denotes the finite set of *choices* for player i,
- $U_i : \times_{j \in I} C_j \to \mathbb{R}$ denotes the *utility function* of player i.

Belief about the opponents' choices

Definition

Let Γ be a static game, and i be a player. A *belief for player i about the opponents' choices* is a probability distribution

$$b_i : C_{-i} \to [0; 1]$$

over the set of opponents' choice-combinations $C_{-i} = \times_{j \in I \setminus \{i\}} C_j$.

Expected utility

Definition

Let Γ be a static game, and i be a player with utility function U_i. Suppose that player i entertains belief b_i and chooses c_i. The *expected utility for player i* is

$$u_i(c_i, b_i) = \sum_{c_{-i} \in C_{-i}} b_i(c_{-i}) \cdot U_i(c_i, c_{-i}),$$

where $(c_i, c_{-i}) = (c_1, \ldots, c_n) \in \times_{j \in I} C_j$.

Optimality

Definition

Let Γ be a static game, and i be a player with utility function U_i. Suppose that player i entertains belief b_i. A choice c_i for player i is *optimal*, iff

$$u_i(c_i, b_i) \geq u_i(c_i', b_i)$$

holds for all choices $c_i' \in C_i$ of player i.

Rationality

Definition

Let Γ be a static game, and i be a player with utility function U_i. A choice c_i for player i is *rational*, iff there exists a belief b_i for player i about the opponents' choices such that c_i is optimal.

Randomizing

Definition

Let Γ be a static game, and i be a player. A *randomized choice* for player i is a probability distribution

$$r_i : C_i \to [0; 1]$$

over the set C_i of player i's choices

Utility with randomizing

Definition

Let Γ be a static game, and i be a player with utility function U_i. Suppose that player i chooses r_i, and that his opponents choose according to c_{-i}. The *randomizing-utility for player i* is

$$V_i(r_i, c_{-i}) = \sum_{c_i \in C_i} r_i(c_i) \cdot U_i(c_i, c_{-i}),$$

where $(c_i, c_{-i}) = (c_1, \ldots, c_n) \in \times_{j \in I} C_j$.

Expected utility with randomizing

Definition

Let Γ be a static game, and i be a player with utility function U_i. Suppose that player i entertains belief b_i and chooses r_i. The *expected randomizing-utility for player i* is

$$v_i(r_i, b_i) = \sum_{c_{-i} \in C_{-i}} b_i(c_{-i}) \cdot V_i(r_i, c_{-i})$$

$$= \sum_{c_{-i} \in C_{-i}} b_i(c_{-i}) \cdot \Big(\sum_{c_i \in C_i} r_i(c_i) \cdot U_i(c_i, c_{-i}) \Big),$$

where $(c_i, c_{-i}) = (c_1, \ldots, c_n) \in \times_{j \in I} C_j$.

Strict Dominance: the pure case

Definition

Let Γ be a static game, and i be a player. A choice c_i for player i is *strictly dominated by another choice*, iff there exists some choice $c_i' \in C_i$ of player i such that

$$U_i(c_i, c_{-i}) < U_i(c_i', c_{-i})$$

holds for every opponents' choice combination $c_{-i} \in C_{-i}$.

Strict Dominance: the randomized case

Definition

Let Γ be a static game, and i be a player. A choice c_i for player i is *strictly dominated by a randomized choice*, iff there exists some randomized choice $r_i \in \Delta(C_i)$ of player i such that

$$U_i(c_i, c_{-i}) < V_i(r_i, c_{-i})$$

holds for every opponents' choice combination $c_{-i} \in C_{-i}$.

Strict Dominance

Definition

Let Γ be a static game, and i be a player. A choice c_i for player i is *strictly dominated*, iff c_i is either strictly dominated by another choice or strictly dominated by a randomized choice.

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A basic lemma

Basic-Lemma I

Let I be some index set, $0 \leq \alpha_i \leq 1$ for all $i \in I$ such that $\sum_{i \in I} \alpha_i = 1$, $x \in \mathbb{R}$, and $y_i \in \mathbb{R}$ for all $i \in I$. If $x < \sum_{i \in I} \alpha_i y_i$, then there exists $i^* \in I$ such that $x < y_{i^*}$.

Proof:

- Towards a contradiction suppose that $x \geq y_i$ for all $i \in I$.
- Then, $\alpha_i x \geq \alpha_i y_i$ holds for all $i \in I$.
- It directly follows that $1 \cdot x = \sum_{i \in I} \alpha_i x \geq \sum_{i \in I} \alpha_i y_i$, a contradiction.

A second basic lemma

Basic-Lemma II

Let I be some index set, $0 < \alpha_i < 1$ for all $i \in I$ such that $\sum_{i \in I} \alpha_i = 1$, $x \in \mathbb{R}$, and $y_i \in \mathbb{R}$ for all $i \in I$. If $x \leq \sum_{i \in I} \alpha_i y_i$, then (there exists $i^* \in I$ such that $x < y_{i^*}$) or ($x = y_i$ for all $i \in I$).

Proof:

- By contraposition, suppose that $x \geq y_i$ for all $i \in I$ and that there exists $i' \in I$ such that $x \neq y_{i'}$.
- Then, $x > y_{i'}$.
- As $0 < \alpha_i < 1$ holds for all $i \in I$, it is the case that $\alpha_{i'} x > \alpha_{i'} y_{i'}$ and $\alpha_i x \geq \alpha_i y_i$ for all $i \in I \setminus \{i'\}$.
- It follows that $x = \sum_{i \in I} \alpha_i x > \sum_{i \in I} \alpha_i y_i$.

Two useful facts

Remark 1

If a choice c_i is strictly dominated by c_i^*,
then $u_i(c_i, b_i) < u_i(c_i^*, b_i)$ for all beliefs $b_i \in \Delta(C_{-i})$.

Proof:

- By definition $U_i(c_i, c_{-i}) < U_i(c_i^*, c_{-i})$ holds for all $c_{-i} \in C_{-i}$.
- Let $b_i \in \Delta(C_{-i})$ be some belief for player i.
- Then,

$$b_i(c_{-i}) \cdot U_i(c_i, c_{-i}) \leq b_i(c_{-i}) \cdot U_i(c_i^*, c_{-i}) \text{ for all } c_{-i} \in C_{-i},$$

  and

$$b_i(c'_{-i}) \cdot U_i(c_i, c'_{-i}) < b_i(c'_{-i}) \cdot U_i(c_i^*, c'_{-i}) \text{ for all } c'_{-i} \in \text{supp}(b_i).$$

- Hence, $u_i(c_i, b_i) = \sum_{c_{-i} \in C_{-i}} b_i(c_{-i}) \cdot U_i(c_i, c_{-i}) < \sum_{c_{-i} \in C_{-i}} b_i(c_{-i}) \cdot U_i(c_i^*, c_{-i}) = u_i(c_i^*, b_i)$

Remark 2

If a choice c_i is strictly dominated by r_i,
then $u_i(c_i, b_i) < v_i(r_i, b_i)$ for all beliefs $b_i \in \Delta(C_{-i})$.

Proof:

- Analogously to the pure case.

Pearce's Lemma

Theorem (Pearce's Lemma)

Let Γ be a static game, i be a player, and c_i be a choice for player i.
c_i is rational, iff, c_i is not strictly dominated.

Proof of the *only if* ($\Rightarrow$) direction ("strictly dominated implies irrational")

- Let c_i^{SD} be a choice of player i that is strictly dominated.

Case 1:

- Suppose that c_i^{SD} is **strictly dominated by another choice** c_i^*.
- Remark 1 then implies that $u_i(c_i^{SD}, b_i) < u_i(c_i^*, b_i)$ holds for all beliefs $b_i \in \Delta(C_{-i})$.
- Hence, there exists no belief $b_i \in \Delta(C_{-i})$ such that c_i^{SD} can be optimal, and c_i^{SD} therefore is irrational.

Case 2:

- Suppose that c_i^{SD} is **strictly dominated by a randomized choice** r_i.
- Remark 2 then implies that $u_i(c_i^{SD}, b_i) < v_i(r_i, b_i)$ holds for all beliefs $b_i \in \Delta(C_{-i})$.

Proof of the *only if* ($\Rightarrow$) direction ("strictly dominated implies irrational")

- Observe that by associativity, commutativity, and distributivity it holds that
$v_i(r_i, b_i) = \sum_{c_{-i} \in C_{-i}} b_i(c_{-i}) \cdot \Big(\sum_{c_i \in C_i} r_i(c_i) \cdot U_i(c_i, c_{-i}) \Big) = \sum_{c_i \in C_i} r_i(c_i) \cdot \Big(\sum_{c_{-i} \in C_{-i}} b_i(c_{-i}) \cdot U_i(c_i, c_{-i}) \Big) = \sum_{c_i \in C_i} r_i(c_i) \cdot u_i(c_i, b_i)$

- Hence, $u_i(c_i^{SD}, b_i) < \sum_{c_i \in C_i} r_i(c_i) \cdot u_i(c_i, b_i)$ holds for all beliefs $b_i \in \Delta(C_{-i})$.

Proof of the *only if* ($\Rightarrow$) direction ("strictly dominated implies irrational")

- Let $b'_i \in \Delta(C_{-i})$ be some belief.

- However, as $0 \leq r_i(c_i) \leq 1$ for all $c_i \in C_i$, the inequality

$$u_i(c_i^{SD}, b'_i) < \sum_{c_i \in C_i} r_i(c_i) \cdot u_i(c_i, b'_i)$$

  implies – by Basic-Lemma I – that there exists some choice $c'_i \in C_i$ such that $u_i(c_i^{SD}, b'_i) < u_i(c'_i, b'_i)$.

- Therefore, c_i^{SD} cannot be optimal given belief b'_i.

- As the belief b'_i has been chosen arbitrarily, c_i^{SD} is irrational.

Proof of the *if* ($\Leftarrow$) direction ("irrational implies strictly dominated")

- Let c_i^{IR} be a choice of player i that is irrational.

Step 1: fixing three basic building blocks d, d^+ and f

- Define functions $d : C_i \times \Delta(C_{-i}) \to \mathbb{R}$ and $d^+ : C_i \times \Delta(C_{-i}) \to \mathbb{R}$ such that

$$d(c_i, b_i) := u_i(c_i, b_i) - u_i(c_i^{IR}, b_i)$$

and

$$d^+(c_i, b_i) := \max\{0, d(c_i, b_i)\}$$

for every choice-belief pair $(c_i, b_i) \in C_i \times \Delta(C_{-i})$ of player i.

Proof of the *if* ($\Leftarrow$) direction ("irrational implies strictly dominated")

- Moreover, define a function $f : \Delta(C_{-i}) \to \mathbb{R}$ such that

$$f(b_i) := \sum_{c_i \in C_i} \left(d^+(c_i, b_i)\right)^2$$

for all $b_i \in \Delta(C_{-i})$.

- As the function f is continuous and its domain $\Delta(C_{-i})$ is compact, it follows with **Weierstrass' extreme value theorem** that the function f attains a minimum, i.e. there exists a belief $b_i^{f-min} \in \Delta(C_{-i})$ such that $f(b_i^{f-min}) \leq f(b_i)$ for all $b_i \in \Delta(C_{-i})$.

Proof of the *if* ($\Leftarrow$) direction ("irrational implies strictly dominated")

Step 2: building a randomized choice r_i^*

- Define numbers

$$r_i^*(c_i) := \frac{d^+(c_i, b_i^{f-min})}{\sum_{c_i' \in C_i} d^+(c_i', b_i^{f-min})}$$

for every choice $c_i \in C_i$ of player i.

- **Remark**: the weight that r_i^* assigns to choices increases in the goodness of the respective choice relative to c_i^{IR}.

- Observe that the numbers $r_i^*(c_i)$ for all $c_i \in C_i$ constitute a **randomized choice** $r_i^* \in \Delta(C_i)$.

Proof of the *if* direction ("irrational implies strictly dominated")

1. Well-definedness of r_i^*:
   - As c_i^{IR} is irrational, it cannot be optimal given belief b_i^{f-min}.
   - Hence, there exists some choice $c_i^* \in C_i$ such that $u_i(c_i^*, b_i^{f-min}) > u_i(c_i^{IR}, b_i^{f-min})$.
   - Thus, $d^+(c_i, b_i^{f-min}) > 0$ for at least some choice $c_i \in C_i$.
   - As, by construction, $d^+(c_i, b_i^{f-min}) \geq 0$ for all $c_i \in C_i$, it follows that $\sum_{c_i' \in C_i} d^+(c_i', b_i^{f-min}) > 0$ and therefore $r_i^*(c_i)$ is well-defined for every $c_i \in C_i$.

2. Since $d^+(c_i, b_i^{f-min}) \geq 0$ for every $c_i \in C_i$, it is the case that $r_i^*(c_i) \geq 0$ for every $c_i \in C_i$.

3. Also, it holds that $\sum_{c_i \in C_i} r_i^*(c_i) = \sum_{c_i \in C_i} \frac{d^+(c_i, b_i^{f-min})}{\sum_{c_i' \in C_i} d^+(c_i', b_i^{f-min})} = 1$.

Proof of the *if* ($\Leftarrow$) direction ("irrational implies strictly dominated")

- Next, it is shown that c_i^{IR} is **strictly dominated** by the randomized choice r_i^*, i.e. $U_i(c_i^{IR}, c_{-i}) < V_i(r_i^*, c_{-i})$ for all $c_{-i} \in C_{-i}$, or equivalently, $V_i(r_i^*, c_{-i}) - U_i(c_i^{IR}, c_{-i}) > 0$ for all $c_{-i} \in C_{-i}$.

- Let $c_{-i}^* \in C_{-i}$ be some opponents' choice-combination.

- Consider the belief $b_i^{c_{-i}^*} \in \Delta(C_{-i})$ of player i that assigns probability-1 to the opponents' choice-combination c_{-i}^*.

Proof of the *if* ($\Leftarrow$) direction ("irrational implies strictly dominated")

Step 3: reformulating strict dominance in terms of d and d^+

- Observe that, $V_i(r_i^*, c_{-i}^*) - U_i(c_i^{IR}, c_{-i}^*) = v_i(r_i^*, b_i^{c_{-i}^*}) - u_i(c_i^{IR}, b_i^{c_{-i}^*})$

$$= \sum_{c_i \in C_i} r_i^*(c_i) \cdot u_i(c_i, b_i^{c_{-i}^*}) - \sum_{c_i \in C_i} r_i^*(c_i) \cdot u_i(c_i^{IR}, b_i^{c_{-i}^*})$$

$$= \sum_{c_i \in C_i} r_i^*(c_i) \cdot \left(u_i(c_i, b_i^{c_{-i}^*}) - u_i(c_i^{IR}, b_i^{c_{-i}^*})\right) = \sum_{c_i \in C_i} r_i^*(c_i) \cdot d(c_i, b_i^{c_{-i}^*})$$

- As $r_i^*(c_i) = \frac{d^+(c_i, b_i^{f-min})}{\sum_{c_i' \in C_i} d^+(c_i', b_i^{f-min})}$ for all $c_i \in C_i$, and as $\sum_{c_i' \in C_i} d^+(c_i', b_i^{f-min}) > 0$, the inequality $V_i(r_i^*, c_{-i}^*) - U_i(c_i^{IR}, c_{-i}^*) > 0$ is **equivalent** to the inequality $\sum_{c_i \in C_i} d^+(c_i, b_i^{f-min}) \cdot d(c_i, b_i^{c_{-i}^*}) > 0$.

Proof of the *if* ($\Leftarrow$) direction ("irrational implies strictly dominated")

Step 4: building a belief b_i^{λ} in terms of the f-minimal belief b_i^{f-min} and the probability-1 belief $b_i^{c^*_{-i}}$

- For every $\lambda \in [0;1]$ define $b_i^{\lambda} := (1-\lambda) \cdot b_i^{f-min} + \lambda \cdot b_i^{c^*_{-i}}$ such that $b_i^{\lambda}(c_{-i}) = (1-\lambda) \cdot b_i^{f-min}(c_{-i}) + \lambda \cdot b_i^{c^*_{-i}}(c_{-i})$ for all $c_{-i} \in C_{-i}$.

- Observe that $b_i^{\lambda} \in \Delta(C_{-i})$ for all $\lambda \in [0;1]$ is indeed a belief for player i. *("a convex combination of two beliefs always is a belief")*
  - Note that for all $\lambda \in [0;1]$, it is the case that $0 \leq b_i^{\lambda}(c_{-i}) \leq 1$ for all $c_{-i} \in C_{-i}$.
  - Note that for all $\lambda \in [0;1]$, it is the case that $\sum_{c_{-i} \in C_{-i}} b_i^{\lambda}(c_{-i})$ $= (1-\lambda) \cdot \sum_{c_{-i} \in C_{-i}} b_i^{f-min}(c_{-i}) + \lambda \cdot \sum_{c_{-i} \in C_{-i}} b_i^{c^*_{-i}}(c_{-i}) = 1$.

Proof of the *if* ($\Leftarrow$) direction ("irrational implies strictly dominated")

Step 5: fixing a small number ϵ to make d negative in b_i^{λ} if so in b_i^{f-min}

- Now, choose a real number $\epsilon > 0$ such that for all $c_i \in C_i$, if $d(c_i, b_i^{f-min}) < 0$, then $d(c_i, b_i^{\lambda}) < 0$ for all $\lambda \in [0; \epsilon]$.
- Observe that such an ϵ exists for all $c_i \in C_i$.
  - Let $c_i \in C_i$ be a choice for player i such that $d(c_i, b_i^{f-min}) < 0$.
  - If $\lambda = 0$, then $b_i^{\lambda} = b_i^{f-min}$ and thus $d(c_i, b_i^{\lambda}) < 0$ immediately holds.
  - Note that $d(c_i, b_i^{\lambda}) = u_i(c_i, b_i^{\lambda}) - u_i(c_i^{IR}, b_i^{\lambda}) =$

    $\sum_{c_{-i} \in C_{-i}} \Big(\big((1-\lambda)b_i^{f-min}(c_{-i}) + \lambda b_i^{c^*_{-i}}(c_{-i})\big) U_i(c_i, c_{-i}) - \big((1-\lambda)b_i^{f-min}(c_{-i}) + \lambda b_i^{c^*_{-i}}(c_{-i})\big) U_i(c_i^{IR}, c_{-i}) \Big)$

    is linear – and hence continuous – in λ.
  - By continuity of $d(c_i, b_i^{\lambda})$ in λ there exists $\epsilon_{c_i} > 0$ such that $d(c_i, b_i^{\lambda}) < 0$ also holds for all $\lambda < \epsilon_{c_i}$.
  - Choose $\epsilon = \min\{\epsilon_{c_i} : c_i \in C_i\}$.

Proof of the *if* ($\Leftarrow$) direction ("irrational implies strictly dominated")

Step 6: establishing an inequality about d^+ and d

- It is shown for all $c_i \in C_i$ that the inequality

$$\left(d^+(c_i, b_i^\lambda)\right)^2 \leq \left((1-\lambda) \cdot d^+(c_i, b_i^{f-min}) + \lambda \cdot d(c_i, b_i^{c^*_{-i}})\right)^2 \qquad (\circ)$$

  holds for all $\lambda \in (0; \epsilon]$.

- Let $c_i^\circ \in C_i$ be some choice for player i and $\lambda^\circ \in (0; \epsilon]$ some "small" positive number.

- **Case 1:** Suppose that $d(c_i^\circ, b_i^{\lambda^\circ}) < 0$. Then, $d^+(c_i^\circ, b_i^{\lambda^\circ}) = 0$, and the inequality $(\circ)$ holds.

Proof of the *if* ($\Leftarrow$) direction ("irrational implies strictly dominated")

Required to show: $(d^+(c_i^\circ, b_i^{\lambda^\circ}))^2 \leq ((1-\lambda^\circ)\cdot d^+(c_i^\circ, b_i^{f-min}) + \lambda^\circ \cdot d(c_i^\circ, b_i^{c^*_{-i}}))^2 \qquad (\circ)$

- **Case 2:** Suppose that $d(c_i^\circ, b_i^{\lambda^\circ}) \geq 0$. The appropriate choice of ϵ assures that $d(c_i^\circ, b_i^{f-min}) \geq 0$, and therefore $d^+(c_i^\circ, b^{\lambda^\circ}) = d(c_i^\circ, b_i^{\lambda^\circ})$ as well as $d^+(c_i^\circ, b_i^{f-min}) = d(c_i^\circ, b_i^{f-min})$.
- Thus, $d^+(c_i^\circ, b_i^{\lambda^\circ}) = d(c_i^\circ, (1-\lambda^\circ)\cdot b_i^{f-min} + \lambda^\circ \cdot b_i^{c^*_{-i}})$.
- As c_i° and λ° are fixed, $d(c_i^\circ, (1-\lambda^\circ)\cdot b_i^{f-min} + \lambda^\circ \cdot b_i^{c^*_{-i}})$ is a linear function in i's beliefs b_i, thus $d(c_i^\circ, (1-\lambda^\circ)\cdot b_i^{f-min} + \lambda^\circ \cdot b_i^{c^*_{-i}}) = (1-\lambda^\circ)\cdot d(c_i^\circ, b_i^{f-min}) + \lambda^\circ \cdot d(c_i^\circ, b_i^{c^*_{-i}})$.
- Consequently, $d^+(c_i^\circ, b_i^{\lambda^\circ}) = (1-\lambda^\circ)\cdot d^+(c_i^\circ, b_i^{f-min}) + \lambda^\circ \cdot d(c_i^\circ, b_i^{c^*_{-i}})$ results, which directly implies the inequality $(\circ)$.
- Hence, $(\circ)$ holds for all $c_i^\circ \in C_i$ and for all $\lambda^\circ \in (0;\epsilon]$.

Proof of the *if* ($\Leftarrow$) direction ("irrational implies strictly dominated")

Step 7: deriving consequences for f in b_i^λ

- Then,

$$f(b_i^\lambda) = \sum_{c_i \in C_i} \left(d^+(c_i, b_i^\lambda)\right)^2$$

$$\leq \sum_{c_i \in C_i} \left((1-\lambda) \cdot d^+(c_i, b_i^{f-min}) + \lambda \cdot d(c_i, b_i^{c^*_{-i}})\right)^2$$

$$= (1-\lambda)^2 \cdot \sum_{c_i \in C_i} \left(d^+(c_i, b_i^{f-min})\right)^2 + 2\lambda(1-\lambda) \cdot \left(\sum_{c_i \in C_i} d^+(c_i, b_i^{f-min}) \cdot d(c_i, b_i^{c^*_{-i}})\right)$$

$$+\lambda^2 \cdot \sum_{c_i \in C_i} \left(d(c_i, b_i^{c^*_{-i}})\right)^2 \text{ for all } \lambda \in (0; \epsilon]$$

- Recall that $f(b_i^{f-min}) \leq f(b_i)$ for all $b_i \in \Delta(C_{-i})$.

Proof of the *if* ($\Leftarrow$) direction ("irrational implies strictly dominated")

- Thus, $\sum_{c_i \in C_i} \left(d^+(c_i, b_i^{f-min})\right)^2 = f(b_i^{f-min}) \leq f(b_i^{\lambda})$

$$\leq (1-\lambda)^2 \cdot \sum_{c_i \in C_i} \left(d^+(c_i, b_i^{f-min})\right)^2 + 2\lambda(1-\lambda) \cdot \Big(\sum_{c_i \in C_i} d^+(c_i, b_i^{f-min}) \cdot d(c_i, b_i^{c^*_{-i}}) \Big)$$

$$+\lambda^2 \cdot \sum_{c_i \in C_i} \left(d(c_i, b_i^{c^*_{-i}})\right)^2 \text{ for all } \lambda \in (0; \epsilon].$$

- It follows for all $\lambda \in (0; \epsilon]$ that $\left(1 - (1-\lambda)^2\right) \sum_{c_i \in C_i} \left(d^+(c_i, b_i^{f-min})\right)^2$

$$= (2\lambda - \lambda^2) \sum_{c_i \in C_i} \left(d^+(c_i, b_i^{f-min})\right)^2$$

$$\leq 2\lambda(1-\lambda) \cdot \Big(\sum_{c_i \in C_i} d^+(c_i, b_i^{f-min}) \cdot d(c_i, b_i^{c^*_{-i}}) \Big) + \lambda^2 \cdot \sum_{c_i \in C_i} \left(d(c_i, b_i^{c^*_{-i}})\right)^2.$$

Proof of the *if* ($\Leftarrow$) direction ("irrational implies strictly dominated")

- Dividing both sides of the inequality by $\lambda > 0$ yields $(2-\lambda)\sum_{c_i \in C_i}\left(d^+(c_i, b_i^{f-min})\right)^2$
$$\leq 2(1-\lambda)\cdot\Big(\sum_{c_i \in C_i} d^+(c_i, b_i^{f-min})\cdot d(c_i, b_i^{c^*_{-i}})\Big) + \lambda\cdot\sum_{c_i \in C_i}\left(d(c_i, b_i^{c^*_{-i}})\right)^2$$
for all $\lambda \in (0;\epsilon]$.
- Let λ approach 0 and obtain
$$\sum_{c_i \in C_i}\left(d^+(c_i, b_i^{f-min})\right)^2 \leq \Big(\sum_{c_i \in C_i} d^+(c_i, b_i^{f-min})\cdot d(c_i, b_i^{c^*_{-i}})\Big)$$
- Recall that $\sum_{c_i \in C_i} d^+(c_i, b_i^{f-min}) > 0$ and thus $\sum_{c_i \in C_i}\left(d^+(c_i, b_i^{f-min})\right)^2 > 0$.
- Therefore, $\sum_{c_i \in C_i} d^+(c_i, b_i^{f-min})\cdot d(c_i, b_i^{c^*_{-i}}) > 0$ obtains.

Proof of the *if* ($\Leftarrow$) direction ("irrational implies strictly dominated")

Step 8: establishing that r_i^* strictly dominates c_i^{IR}

- Recall that
$$\sum_{c_i \in C_i} d^+(c_i, b_i^{f-min}) \cdot d(c_i, b_i^{c^*_{-i}}) > 0$$
is equivalent to
$$V_i(r_i^*, c^*_{-i}) > U_i(c_i^{IR}, c^*_{-i}).$$

- As the opponents' choice combination c^*_{-i} has been chosen arbitrarily, it can be concluded that $U_i(c_i^{IR}, c_{-i}) < V_i(r_i^*, c_{-i})$ holds for all $c_{-i} \in C_{-i}$, and the irrational choice c_i^{IR} is thus strictly dominated by the randomized choice r_i^*.

Agenda

- Introduction
- Definitions
- Proof
- **Appendix: Weierstrass' extreme value theorem**

Topology, topological space, and open sets

Definition

A topology on some set X is a set $\mathcal{T} \subseteq \mathcal{P}(X)$ of subsets of X such that

- $\emptyset, X \in \mathcal{T}$,
- if $T, T' \in \mathcal{T}$, then $T \cap T' \in \mathcal{T}$,
- if $T_i \in \mathcal{T}$ for all $i \in I$, then $\cup_{i \in I} T_i \in \mathcal{T}$.

A set X for which a topology $\mathcal{T}$ has been specified is called a topological space. A set $T \in \mathcal{T}$ is called open set.

Standard topology

Definition

A set $O \subseteq \mathbb{R}$ is called open, if for all $o \in O$ there exists $\epsilon > 0$ such that $(o - \epsilon; o + \epsilon) \subseteq O$. The set containing all such sets O is called standard topology of $\mathbb{R}$.

Open sets in $\mathbb{R}$ with the standard topology

Remark

Let $a, b \in \mathbb{R}$ and $\mathbb{R}$ be equipped with the standard topology. The open interval $(a; b)$ is an open set.

Argument:

- Let $x \in (a; b)$ and $\epsilon < \min\{| x - a |, | b - x |\}$.
- Then, $(x - \epsilon; x + \epsilon) \subseteq (a; b)$.
- Therefore, $(a; b)$ is open.

Remark

Let $a \in \mathbb{R}$ and $\mathbb{R}$ be equipped with the standard topology. The open intervals $(a; +\infty)$ and $(-\infty; a)$ are open sets.

Argument:

- Note that $(a; +\infty) = \cup_{r>a}(a; r)$ and that $(-\infty; a) = \cup_{r<a}(r; a)$.
- As unions of open sets $(a; +\infty)$ and $(-\infty; a)$ are therefore open sets.

Continuity

Definition

Let X and Y be topological spaces with topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$, respectively. A function $X \to Y$ is continuous, if for every open set $V \in \mathcal{T}_Y$, the set $f^{-1}(V) = \{x \in X : f(x) \in V\} \in \mathcal{T}_X$ is open.

Covers

Definition

Let X be a topological space. A set $\mathcal{C} \subseteq \mathcal{P}(X)$ is a cover of X, if the union of the elements of $\mathcal{C}$ is a superset of X. If all elements of $\mathcal{C}$ are open, then $\mathcal{C}$ is called open cover of X.

Compactness

Definition

Let X be a topological space. The space X is compact, if every open cover of X contains a finite number of sets that also cover X.

Continuity preserves compactness

Theorem

Let X and Y be topological spaces, and $f : X \to Y$ be a function. If X is compact and f is continuous, then the image $f(X)$ is compact.

Proof:

- Let $\mathcal{C}$ be an open cover of $f(X)$.
- Note that every $C \in \mathcal{C}$ is open in Y.
- As $\mathcal{C}$ covers Y and $f(X) \subseteq Y$, it follows that $X \subseteq \{f^{-1}(C) : C \in \mathcal{C}\}$, i.e. $\{f^{-1}(C) : C \in \mathcal{C}\}$ covers X.
- Continuity of f ensures that every such set $f^{-1}(C)$ is open in X.
- By compactness of X a finite number of these sets, say $f^{-1}(C_1), \dots, f^{-1}(C_n)$, cover X.
- Then, the sets $C_1, \dots, C_n$ cover $f(X)$.

Weierstrass' extreme value theorem

Theorem (Weierstrass' extreme value theorem)

Let X be a compact topological space, and $f : X \to \mathbb{R}$ be a continuous function, where $\mathbb{R}$ is equipped with the standard topology. Then, there exist $a, b \in X$ such that $f(a) \leq f(x) \leq f(b)$ for all $x \in X$.

Proof of Weierstrass' extreme value theorem

Proof:

- Since X is compact and f is continuous, the image $f(X)$ is compact.
- Suppose $f(X)$ has no smallest element, i.e. there exists no $m \in f(X)$ such that $m \leq y$ for all $y \in f(X)$.
- Then, the set $\{(y; +\infty) : y \in f(X)\}$ forms an open cover of $f(X)$.
- By compactness of $f(X)$ a finite number of these sets, say $(y_i; +\infty), \ldots, (y_n; +\infty)$ cover $f(X)$, and consider $\min\{y_1, \ldots, y_n\}$.
- Note that $\min\{y_1, \ldots, y_n\} \leq y$ for all $y \in f(X)$, a contradiction.
- As $\min\{y_1, \ldots, y_n\} \in f(X)$ there exists $a \in X$ such that $f(a) = \min\{y_1, \ldots, y_n\}$.
- Analogously for b.

Thank you!