# Lexicographic Beliefs

## Part II: Respect of Preferences

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# Introduction

- Cautious reasoning = not completely discarding any event, yet being able to consider some event much more likely, indeed infinitely more likely, than some other event
- Modelling tool: lexicographic beliefs
- A particular way of cautious reasoning is based on primary belief in rationality: restrictions concentrate mainly on the first lexicographic level
- However, it can also be plausible to impose conditions on deeper lexicographic levels!

# Agenda

- Respecting the Opponent's Preferences
- Common Full Belief in (Caution & Respect of Preferences)
- Existence
- Towards an Algorithm
- Algorithm

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# Taking the Opponent's Preferences Seriously

**Motivating Idea:**

- If player $i$ believes that his opponent $j$ prefers some choice $c_j$ to some other choice $c_j'$, then he must deem $c_j$ infinitely more likely than $c_j'$.

# Motivating Example: Where to read my book?

**Story**

- *You* would like to go to a pub to read your book.
- *Barbara* is going to a pub as well, but *you* forgot to ask her to which one.
- Your only objective is to avoid *Barbara*, since *you* would like to read your book in silence.
- *Barbara* prefers *Pub A* to *Pub B*, and *Pub B* to *Pub C*.
- Question: Which pub should *you* go to?

# Motivating Example: Where to read my book?

| | | Barbara | | |
|---|---|---|---|---|
| | | $A$ | $B$ | $C$ |
| | $A$ | 0, 3 | 1, 2 | 1, 1 |
| You | $B$ | 1, 3 | 0, 2 | 1, 1 |
| | $C$ | 1, 3 | 1, 2 | 0, 1 |

# Motivating Example: Where to read my book?

| | | Barbara $A$ | Barbara $B$ | Barbara $C$ |
|---|---|---|---|---|
| You | $A$ | $0, 3$ | $1, 2$ | $1, 1$ |
| | $B$ | $1, 3$ | $0, 2$ | $1, 1$ |
| | $C$ | $1, 3$ | $1, 2$ | $0, 1$ |

- **Type Spaces:** $T_{you} = \{t_y\}$ and $T_{Barbara} = \{t_B\}$
- **Beliefs for** *You*: $b_{you}^{lex}(t_y) = ((A, t_B); (C, t_B); (B, t_B))$
- **Beliefs for** *Barbara*: $b_{Barbara}^{lex}(t_B) = ((B, t_y); (C, t_y); (A, t_y))$

- Your type $t_y$ primarily believes in Barbara's rationality.
- However, $t_y$'s secondary and tertiary belief seem counter-intuitive.
- For Barbara, *B* is better than *C*, hence it can be plausible to deem Barbara choosing *B* infinitely more likely than her picking *C*.

# Respecting the Opponent's Preferences

## Definition

A cautious type $t_i$ of player $i$ ***respects the opponent's preferences***, whenever for every opponent's type $t_j$ deemed possible by $t_i$, if $t_j$ prefers some choice $c_j$ to some other choice $c_j'$, then $t_i$ deems $(c_j, t_j)$ infinitely more likely than $(c_j', t_j)$.

**Intuition:**
A player deems better choices of his opponent infinitely more likely than worse choices.

**Remark:**
Respect of preferences can only be defined for cautious types.

# Example: Where to read my book?

| | | Barbara $A$ | Barbara $B$ | Barbara $C$ |
|---|---|---|---|---|
| You | $A$ | 0, 3 | 1, 2 | 1, 1 |
| | $B$ | 1, 3 | 0, 2 | 1, 1 |
| | $C$ | 1, 3 | 1, 2 | 0, 1 |

- **Type Spaces:** $T_{you} = \{t_y, t_y'\}$ and $T_{Barbara} = \{t_B\}$
- **Beliefs for You:** $b_{you}^{lex}(t_y) = ((A, t_B); (C, t_B); (B, t_B))$ and $b_{you}^{lex}(t_y') = ((A, t_B); (B, t_B); (C, t_B))$
- **Beliefs for Barbara:** $b_{Barbara}^{lex}(t_B) = ((B, t_y); (C, t_y); (A, t_y))$

- Your type $t_y$ does not respect Barbara's preferences.
- Your type $t_y'$ does respect Barbara's preferences.
- Note that if you respect Barbara's preferences, then your unique optimal choice is *C*.

# Respect of Preferences and Primary Belief in Rationality

**Observation.** If $Alice$ is cautious and respects $Bob$'s preferences, then she also primarily believes in $Bob$'s rationality.

- Suppose that $t_{Alice}$ is cautious and respects $Bob$'s preferences.
- Now, consider some pair $(c_{Bob}, t_{Bob})$ that is deemed possible by $t_{Alice}$ such that $c_{Bob}$ is not optimal for $t_{Bob}$.
- Then, there exists some choice $c^*_{Bob}$ that $t_{Bob}$ prefers to $c_{Bob}$, and $t_{Alice}$ must deem $(c^*_{Bob}, t_{Bob})$ infinitely more likely than $(c_{Bob}, t_{Bob})$.
- Thus, $t_{Alice}$'s primary belief must assign probability-0 to $(c_{Bob}, t_{Bob})$.

# Agenda

- Respecting the Opponent's Preferences
- **Common Full Belief in (Caution & Respect of Preferences)**
- Existence
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# Common Full Belief in (Caution & Respect of Preferences)

## Definition

A cautious type $t_i$ of player $i$ expresses ***common full belief in (caution & respect of preferences)***, if

- $t_i$ expresses 1-fold full belief in caution and respect of preferences, i.e. $t_i$ only deems possible cautious opponent $j$'s types and respects $j$'s preferences,
- $t_i$ expresses 2-fold full belief in caution and respect of preferences, i.e. $t_i$ only deems possible opponent $j$'s types that only deem possible cautious $i$'s types and that respect $i$'s preferences,
- etc.

# Relation to Common Full Belief in (Caution & Primary Belief in Rationality)

## Proposition

*If a cautious type $t_i$ expresses common full belief in (caution & respect of preferences), then $t_i$ entertains common full belief in (caution & primary belief in rationality).*

# Example: Where to read my book?

| | | Barbara A | Barbara B | Barbara C |
|---|---|---|---|---|
| You | A | 0, 3 | 1, 2 | 1, 1 |
| You | B | 1, 3 | 0, 2 | 1, 1 |
| You | C | 1, 3 | 1, 2 | 0, 1 |

- **Type Spaces:** $T_{you} = \{t_y\}$ and $T_{Barbara} = \{t_B\}$
- **Beliefs for You:** $b_{you}^{lex}(t_y) = ((A, t_B); (B, t_B); (C, t_B))$
- **Beliefs for Barbara:** $b_{Barbara}^{lex}(t_B) = ((C, t_y); (B, t_y); (A, t_y))$

- Your type $t_y$ is cautious, and respects Barbara's preferences.
- Barbara's type $t_B$ is cautious, and respects your preferences.
- Thus, $t_y$ expresses common full belief in caution and respect of preferences.
- As choice $C$ is optimal for type $t_y$, you can rationally and cautiously go to Pub $C$ under common full belief in (caution & respect of preferences).
- Note that under common full belief in (caution & primary belief in rationality), you can rationally and cautiously choose $B$ as well as $C$.

# Example: Dividing a Pizza

**Story**

- *You* have ordered a four-sliced pizza with *Barbara*.
- Both simultaneously write down the desired number of slices or simply "the rest".
- It is agreed that if the numbers' sum exceeds four, both will give the pizza to charity and neither gets any slice.
- If both write "the rest", then the pizza is divided equally among the two.

# Example: Dividing a Pizza

| You \ Barbara | 0 | 1 | 2 | 3 | 4 | *rest* |
|---|---|---|---|---|---|---|
| 0 | 0, 0 | 0, 1 | 0, 2 | 0, 3 | 0, 4 | 0, 4 |
| 1 | 1, 0 | 1, 1 | 1, 2 | 1, 3 | 0, 0 | 1, 3 |
| 2 | 2, 0 | 2, 1 | 2, 2 | 0, 0 | 0, 0 | 2, 2 |
| 3 | 3, 0 | 3, 1 | 0, 0 | 0, 0 | 0, 0 | 3, 1 |
| 4 | 4, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 4, 0 |
| *rest* | 4, 0 | 3, 1 | 2, 2 | 1, 3 | 0, 4 | 2, 2 |

# Example: Dividing a Pizza

| You \ Barbara | 0 | 1 | 2 | 3 | 4 | *rest* |
|---|---|---|---|---|---|---|
| 0 | 0, 0 | 0, 1 | 0, 2 | 0, 3 | 0, 4 | 0, 4 |
| 1 | 1, 0 | 1, 1 | 1, 2 | 1, 3 | 0, 0 | 1, 3 |
| 2 | 2, 0 | 2, 1 | 2, 2 | 0, 0 | 0, 0 | 2, 2 |
| 3 | 3, 0 | 3, 1 | 0, 0 | 0, 0 | 0, 0 | 3, 1 |
| 4 | 4, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 4, 0 |
| *rest* | 4, 0 | 3, 1 | 2, 2 | 1, 3 | 0, 4 | 2, 2 |

- What choices can you rationally and cautiously make under common full belief in (caution & respect of preferences)?
- Your choices 0, 1, and 2 are weakly dominated by claiming the *rest*.
- Hence, if you are cautious, then the *rest* is better for you than 0, 1, or 2.
- Similarly, if you believe Barbara to be cautious, then you believe the *rest* to be better for her than 0, 1, or 2.
- As you respect Barbara's preferences, you deem her choice *rest* infinitely more likely than 0, 1, and 2.
- It is now shown that 4 is then better for you than 3.

# Example: Dividing a Pizza

| You \ Barbara | 0 | 1 | 2 | 3 | 4 | *rest* |
|---|---|---|---|---|---|---|
| 0 | 0, 0 | 0, 1 | 0, 2 | 0, 3 | 0, 4 | 0, 4 |
| 1 | 1, 0 | 1, 1 | 1, 2 | 1, 3 | 0, 0 | 1, 3 |
| 2 | 2, 0 | 2, 1 | 2, 2 | 0, 0 | 0, 0 | 2, 2 |
| 3 | 3, 0 | 3, 1 | 0, 0 | 0, 0 | 0, 0 | 3, 1 |
| 4 | 4, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 4, 0 |
| *rest* | 4, 0 | 3, 1 | 2, 2 | 1, 3 | 0, 4 | 2, 2 |

- Indeed, suppose that you deem Barbara's choice *rest* infinitely more likely than 0, 1, and 2.
- There are four possible ways to do so:

  1. You deem *rest* infinitely more likely than her other choices. Then, 4 is better for you than 3.
  2. You deem 4 and *rest* infinitely more likely than her other choices. Then, 4 is better for you than 3.
  3. You deem 3 and *rest* infinitely more likely than her other choices. Then, 4 is better for you than 3.
  4. You deem 3, 4 and *rest* infinitely more likely than her other choices. Then, 4 is better for you than 3.

- Thus, if you are cautious, believe in Barbara's caution, and respect Barbara's preferences, then **you prefer** *rest* **to 0, 1, and 2** and **you prefer 4 to 3**.
- Consequently, under common full belief in (caution & respect of preferences) only 4 and *rest* can possibly be optimal for you!

# Example: Dividing a Pizza

| | | Barbara | | | | | |
|---|---|---|---|---|---|---|---|
| | | 0 | 1 | 2 | 3 | 4 | *rest* |
| You | 0 | 0, 0 | 0, 1 | 0, 2 | 0, 3 | 0, 4 | 0, 4 |
| | 1 | 1, 0 | 1, 1 | 1, 2 | 1, 3 | 0, 0 | 1, 3 |
| | 2 | 2, 0 | 2, 1 | 2, 2 | 0, 0 | 0, 0 | 2, 2 |
| | 3 | 3, 0 | 3, 1 | 0, 0 | 0, 0 | 0, 0 | 3, 1 |
| | 4 | 4, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 4, 0 |
| | *rest* | 4, 0 | 3, 1 | 2, 2 | 1, 3 | 0, 4 | 2, 2 |

- Consider the following **lexicographic epistemic model**:
  - **Type Spaces:**
    $T_{you} = \{t_y^4, t_y^r\}$ and $T_{Barbara} = \{t_B^4, t_B^r\}$
  - **Beliefs for *You*:**
    $b_{you}^{lex}(t_y^4) = ((rest, t_B^r); (1, t_B^r); (4, t_B^r); (3, t_B^r); (2, t_B^r); (0, t_B^r))$
    $b_{you}^{lex}(t_y^r) = ((4, t_B^4); (3, t_B^4); (rest, t_B^4); (2, t_B^4); (1, t_B^4); (0, t_B^4))$
  - **Beliefs for *Barbara*:**
    $b_B^{lex}(t_B^4) = ((rest, t_y^r); (1, t_y^r); (4, t_y^r); (3, t_y^r); (2, t_y^r); (0, t_y^r))$
    $b_B^{lex}(t_B^r) = ((4, t_y^4); (3, t_y^4); (rest, t_y^4); (2, t_y^4); (1, t_y^4); (0, t_y^4))$
- Both your types are cautious and express common full belief in (caution & respect of preferences).
- As 4 is optimal for $t_y^4$ and *rest* is optimal for $t_y^r$, you can rationally as well as cautiously choose 4 and *rest* under common full belief in (caution & respect of preferences)!

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# An Important Question

Is it always possible – for any given game – that a player cautiously reasons in line with common full belief in (caution & respect of preferences)?

# Example: Hide and Seek

## Story

- *You* would like to go to a pub to read your book.
- *Barbara* is going to a pub as well, but *you* forgot to ask her to which one.
- Your only objective is to avoid *Barbara*, since *you* would like to read your book in silence.
- *Barbara* prefers *Pub A* to *Pub B*, and *Pub B* to *Pub C*, and she would also like to talk to *you*.
- Question: Which pub should *you* go to?

# Example: Hide and Seek

| | | Barbara | | |
|---|---|---|---|---|
| | | $A_B$ | $B_B$ | $C_B$ |
| | $A_y$ | 0, 5 | 1, 2 | 1, 1 |
| You | $B_y$ | 1, 3 | 0, 4 | 1, 1 |
| | $C_y$ | 1, 3 | 1, 2 | 0, 3 |

# Example: Hide and Seek

| | | Barbara | | |
|---|---|---|---|---|
| | | $A_B$ | $B_B$ | $C_B$ |
| You | $A_y$ | 0, 5 | 1, 2 | 1, 1 |
| | $B_y$ | 1, 3 | 0, 4 | 1, 1 |
| | $C_y$ | 1, 3 | 1, 2 | 0, 3 |

**Is common full belief in (caution & respect of preferences) possible in this game?**

- Consider some arbitrary cautious lexicographic belief about Barbara's choice, e.g. $(A_B; B_B; C_B)$.
- Given this belief, your preferences are $(C_y; B_y; A_y)$.
- Consider a cautious lexicographic belief for Barbara that respects these preferences, e.g. $(C_y; B_y; A_y)$.
- Given this belief, Barbara's preferences are $(A_B; C_B; B_B)$.
- Consider a cautious lexicographic belief for you that respects these preferences, e.g. $(A_B; C_B; B_B)$.
- Given this belief, your preferences are $(B_y; C_y; A_y)$.
- Consider a cautious lexicographic belief for Barbara that respects these preferences, e.g. $(B_y; C_y; A_y)$.
- Given this belief, Barbara's preferences are $(B_B; A_B; C_B)$.
- Consider a cautious lexicographic belief for you that respects these preferences, e.g. $(B_B; A_B; C_B)$.
- Given this belief, your preferences are $(C_y; A_y; B_y)$.
- Consider a cautious lexicographic belief for Barbara that respects these preferences, e.g. $(C_y; A_y; B_y)$.
- Given this belief, Barbara's preferences are $(A_B; C_B; B_B)$.
- Consider a cautious lexicographic belief for you that respects these preferences, e.g. $(A_B; C_B; B_B)$.

# Example: Hide and Seek

| | | Barbara $A_B$ | $B_B$ | $C_B$ |
|---|---|---|---|---|
| You | $A_y$ | 0, 5 | 1, 2 | 1, 1 |
| | $B_y$ | 1, 3 | 0, 4 | 1, 1 |
| | $C_y$ | 1, 3 | 1, 2 | 0, 3 |

- A **sequence of lexicographic beliefs** has thus been formed:
  $(A_B; B_B; C_B) \rightarrow (C_y; B_y; A_y) \rightarrow (A_B; C_B; B_B) \rightarrow (B_y; C_y; A_y) \rightarrow (B_B; A_B; C_B) \rightarrow (C_y; A_y; B_y) \rightarrow (A_B; C_B; B_B)$
- It has entered into a **cylce**:
  $(A_B; C_B; B_B) \rightarrow (B_y; C_y; A_y) \rightarrow (B_B; A_B; C_B) \rightarrow (C_y; A_y; B_y) \rightarrow (A_B; C_B; B_B)$
- This cycle is now transformed into a lexicographic epistemic model.
- **Type Spaces:** $T_{you} = \{t_y, t'_y\}$ and $T_{Barbara} = \{t_B, t'_B\}$
- **Beliefs for** *You***:** $b_y^{lex}(t_y) = ((A_B, t_B); (C_B, t_B); (B_B, t_B))$ and $b_y^{lex}(t'_y) = ((B_B, t'_B); (A_B, t'_B); (C_B, t'_B))$
- **Beliefs for** *Barbara***:** $b_B^{lex}(t_B) = ((C_y, t'_y); (A_y, t'_y); (B_y, t'_y))$ and $b_B^{lex}(t'_B) = ((B_y, t_y); (C_y, t_y); (A_y, t_y))$
- All types in the epistemic model are cautious and respect the opponent's preferences.
- Hence, all express common full belief in (caution & respect of preferences).
- Concluding, caution and common full belief in (caution & respect of preferences) is indeed possible in the *Hide and Seek* game.

# Generalizing the Construction for Existence

- Fix some finite game and consider an arbitrary cautious lexicographic belief $b_i^{lex\,1}$ for player $i$ about $j$'s choice.
- Let $R_i^1$ be the induced preference relation on $C_i$ for player $i$ given this belief.
- Consider some cautious lexicographic belief $b_j^{lex\,2}$ for player $j$ about $i$'s choice that respects the preference relation $R_i^1$.
- Let $R_j^2$ be the induced preference relation on $C_j$ for player $j$ given this belief.
- Consider some cautious lexicographic belief $b_i^{lex\,3}$ for player $i$ about $j$'s choice that respects the preference relation $R_i^2$.
- Let $R_i^3$ be the induced preference relation on $C_j$ for player $j$ given this belief.
- etc.
- The sequence of lexicographic beliefs thus constructed bears the following property: Any element of the sequence satisfies respect of preferences given the preference relation induced by the immediate predecessor lexicographic belief in the sequence.
- Since there are only finitely many choices and the same lexicographic belief can be specified for any recurring preference relation, the sequence of lexicographic beliefs must eventually enter into a cycle of lexicographic beliefs.

# From Lexicographic Beliefs to Types

- Suppose some cycle of lexicographic beliefs:
  $b_i^{lex\,1} \to b_j^{lex\,2} \to b_i^{lex\,3} \to \ldots \to b_j^{lex\,K} \to b_i^{lex\,1}$

- This cycle can be transformed into an lexicographic epistemic model:
  - $b_i(t_i^1) = (b_i^{lex\,1}, t_j^K)$
  - $b_j(t_j^2) = (b_j^{lex\,2}, t_i^1)$
  - $b_i(t_i^3) = (b_i^{lex\,3}, t_j^2)$
  - $b_j(t_j^4) = (b_j^{lex\,4}, t_i^3)$
  - etc.

- In such an epistemic model, every type is cautious and respects the opponent's preferences.

- Hence, all types express common full belief in (caution & respect of preferences)!

# Existence

**Theorem**

*Let $\Gamma$ be some finite two player game. Then, there exists a lexicographic epistemic model such that*

- *every type in the model is cautious and expresses common full belief in (caution & respect of preferences),*
- *every type in the model deems possible only one opponent's type, and assigns at each lexicographic level probability-1 to one of the opponent's choices.*

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# Towards an Algorithm: Elimination of Choices?

- It is very convenient to have an algorithm which computes the choices that can be made rationally under caution and common full belief in (caution & respect of preferences).
- So far algorithms have been presented that iteratively eliminate choices from the game.
- It is now shown that such an algorithm cannot work for common full belief in (caution & respect of preferences).

# Example: Spy Game

## Story

- *You* would like to go to a pub to read your book.
- *Barbara* is going to a pub as well, but *you* forgot to ask her to which one.
- Your only objective is to avoid *Barbara*, since *you* would like to read your book in silence.
- *Barbara* prefers *Pub A* to *Pub B*, and *Pub B* to *Pub C*.
- Besides, *Barbara* suspects *you* to have an affair and would thus like to spy on you.
- Spying is only possible from *Pub A* to *Pub C*, or vice versa.
- *Barbara* derives additional utility of 3 from spying.
- Question: Which pub should *you* go to?

# Example: Spy Game

|  |  | Barbara |  |  |
|---|---|---|---|---|
|  |  | $A_B$ | $B_B$ | $C_B$ |
| You | $A_y$ | 0, 3 | 1, 2 | 1, 4 |
|  | $B_y$ | 1, 3 | 0, 2 | 1, 1 |
|  | $C_y$ | 1, 6 | 1, 2 | 0, 1 |

# Example: Spy Game

| | | Barbara | | |
|---|---|---|---|---|
| | | $A_B$ | $B_B$ | $C_B$ |
| You | $A_y$ | 0, 3 | 1, 2 | 1, 4 |
| | $B_y$ | 1, 3 | 0, 2 | 1, 1 |
| | $C_y$ | 1, 6 | 1, 2 | 0, 1 |

- Which pubs can you rationally and cautiously pick under common full belief in (caution & respect of preferences)?
- Barbara prefers $A_B$ to $B_B$.
- Therefore, you must deem $A_B$ infinitely more likely than $B_B$.
- Then, you prefer $B_y$ to $A_y$.
- Hence, you believe that Barbara deems $B_y$ infinitely more likely than $A_y$.
- Thus, you believe that Barbara prefers $B_B$ to $C_B$.
- Consequently, you must deem Barbara's choice $B_B$ infinitely more likely than $C_B$.
- As you deem $A_B$ infinitely more likely than $B_B$ and $B_B$ infinitely more likely than $C_B$, you can only rationally choose $C_y$!

# Example: Spy Game

| | | Barbara | | |
|---|---|---|---|---|
| | | $A_B$ | $B_B$ | $C_B$ |
| You | $A_y$ | 0, 3 | 1, 2 | 1, 4 |
| | $B_y$ | 1, 3 | 0, 2 | 1, 1 |
| | $C_y$ | 1, 6 | 1, 2 | 0, 1 |

- Consider the following **lexicographic epistemic model**:
  - **Type Spaces:**

    $T_{you} = \{t_y\}$ and $T_{Barbara} = \{t_B\}$
  - **Beliefs for *You*:**

    $b_{you}(t_y) = ((A_B, t_B); (B_B, t_B); (C_B, t_B))$
  - **Beliefs for *Barbara*:**

    $b_{Barbara}(t_B) = ((C_y, t_y); (B_y, t_y); (A_y, t_y))$
- Both your types are cautious and express common full belief in (caution & respect of preferences).
- As $C_y$ is optimal for $t_y$, you can indeed rationally and cautiously choose $C_y$ under common full belief in (caution & respect of preferences)!

# Example: Spy Game

| | | Barbara | | |
|---|---|---|---|---|
| | | $A_B$ | $B_B$ | $C_B$ |
| You | $A_y$ | 0, 3 | 1, 2 | 1, 4 |
| | $B_y$ | 1, 3 | 0, 2 | 1, 1 |
| | $C_y$ | 1, 6 | 1, 2 | 0, 1 |

- **But:** choice $C_y$ cannot be uniquely filtered out by iteratively deleting strictly or weakly dominated choices!
- At a first step, only $B_B$ could be eliminated.
- But then choice $B_y$ could never be eliminated in the resulting reduced game!

# Likelihood Orderings

## Definition

A ***likelihood ordering*** for player $i$ on $j$'s choice set is a sequence $L_i = (L_i^1; L_i^2; \ldots; L_i^K)$, where $\{L_i^1; L_i^2; \ldots; L_i^K\}$ forms a partition of $C_j$.

**Interpretation:**

- Player $i$ deems all choices in $L_i^1$ infinitely more likely than all choices in $L_i^2$; deems all choices in $L_i^2$ infinitely more likely than all choices in $L_i^3$; etc.
- Moreover, a likelihood ordering $L_i$ for player $i$ is said to **assume** a set of choices $D_j$ for the opponent $j$, whenever $L_i$ deems all choices inside $D_j$ infinitely more likely than all choices outside $D_j$.
- In other words, an assumed set of choices equals the union of some first $l$ levels of a likelihood ordering.

# Preference Restrictions

**Definition**

A ***preference restriction*** for player $i$ is a pair $(c_i, A_i)$, where $c_i \in C_i$ and $A_i \subseteq C_i$.

**Interpretation:**

- Player $i$ "prefers" at least one choice in $A_i$ to $c_i$.
(Note that "prefer" is used intuitively here, it does not correspond to the well-defined notion prefer!)
- Besides, a likelihood ordering $L_i$ for player $i$ is said to **respect a preference restriction** $(c_j, A_j)$ for the opponent $j$, whenever $L_i$ deems at least one choice in $A_j$ infinitely more likely than $c_j$

# Example: Spy Game

## Story

- *You* would like to go to a pub to read your book.
- *Barbara* is going to a pub as well, but *you* forgot to ask her to which one.
- Your only objective is to avoid *Barbara*, since *you* would like to read your book in silence.
- *Barbara* prefers *Pub A* to *Pub B*, and *Pub B* to *Pub C*.
- Besides, *Barbara* suspects you to have an affair and would thus like to spy on *you*.
- Spying is only possible from *Pub A* to *Pub C*, or vice versa.
- *Barbara* derives additional utility of 3 from spying.
- Question: Which pub should *you* go to?

# Example: Spy Game

| | | Barbara $A_B$ | Barbara $B_B$ | Barbara $C_B$ |
|---|---|---|---|---|
| You | $A_y$ | 0, 3 | 1, 2 | 1, 4 |
| You | $B_y$ | 1, 3 | 0, 2 | 1, 1 |
| You | $C_y$ | 1, 6 | 1, 2 | 0, 1 |

# Example: Spy Game

| | | Barbara $A_B$ | $B_B$ | $C_B$ |
|---|---|---|---|---|
| You | $A_y$ | 0, 3 | 1, 2 | 1, 4 |
| | $B_y$ | 1, 3 | 0, 2 | 1, 1 |
| | $C_y$ | 1, 6 | 1, 2 | 0, 1 |

- Barbara prefers $A_B$ to $B_B$.
- It has been shown above that eliminating choice $B_B$ leads to a dead end.
- However, it can be noted that $(B_B, \{A_B\})$ is a preference restriction for Barbara.
- If you respect Barbara's preference restriction $(B_B, \{A_B\})$, then you must deem $A_B$ infinitely more likely than $B_B$.
- Thus, your likelihood ordering should be one of the followng:

  1. $(\{A_B\}, \{B_B\}, \{C_B\})$
  2. $(\{A_B\}, \{C_B\}, \{B_B\})$
  3. $(\{A_B\}, \{B_B, C_B\})$
  4. $(\{C_B\}, \{A_B\}, \{B_B\})$
  5. $(\{A_B, C_B\}, \{B_B\})$

- If your likelihood ordering is $(\{A_B\}, \{B_B\}, \{C_B\})$ or $(\{A_B\}, \{C_B\}, \{B_B\})$ or $(\{A_B\}, \{B_B, C_B\})$, then you assume Barbara's choice $A_B$, i.e. you deem $A_B$ infinitely more likely than her other choices.
- In this case, you prefer $B_y$ to $A_y$, since $B_y$ weakly dominates $A_y$ on $\{A_B\}$.

# Example: Spy Game

| | | Barbara | | |
|---|---|---|---|---|
| | | $A_B$ | $B_B$ | $C_B$ |
| You | $A_y$ | $0, 3$ | $1, 2$ | $1, 4$ |
| | $B_y$ | $1, 3$ | $0, 2$ | $1, 1$ |
| | $C_y$ | $1, 6$ | $1, 2$ | $0, 1$ |

- Your likelihood ordering should be one of the followng:

  1. $(\{A_B\}, \{B_B\}, \{C_B\})$
  2. $(\{A_B\}, \{C_B\}, \{B_B\})$
  3. $(\{A_B\}, \{B_B, C_B\})$
  4. $(\{C_B\}, \{A_B\}, \{B_B\})$
  5. $(\{A_B, C_B\}, \{B_B\})$

- If your likelihood ordering is $(\{C_B\}, \{A_B\}, \{C_B\})$ or $(\{A_B, C_B\}, \{B_B\})$, then you assume Barbara's choice set $\{A_B, C_B\}$, i.e. you deem $A_B$ and $C_B$ infinitely more likely than her choice $B_B$.
- In this case, you prefer $B_y$ to $A_y$, since $B_y$ weakly dominates $A_y$ on $\{A_B, C_B\}$.
- Indeed, every likelihood ordering for you that respects Barbara's preference restriction $(B_B, \{A_B\})$ assumes either $\{A_B\}$ or $\{A_B, C_B\}$, and on both sets your choice $A_y$ is weakly dominated by $B_y$.
- Hence, Barbara's preference restriction $(B_B, \{A_B\})$ induces the new preference restriction $(A_y, \{B_y\})$ for you.

# Example: Spy Game

| | | Barbara | | |
|---|---|---|---|---|
| | | $A_B$ | $B_B$ | $C_B$ |
| You | $A_y$ | 0, 3 | 1, 2 | 1, 4 |
| | $B_y$ | 1, 3 | 0, 2 | 1, 1 |
| | $C_y$ | 1, 6 | 1, 2 | 0, 1 |

- So far there are two preference restrictions: $(A_y, \{B_y\})$ and $(B_B, \{A_B\})$.
  - If Barbara respects your preference restriction $(A_y, \{B_y\})$, then she must deem $B_y$ infinitely more likely than $A_y$.
  - Hence, her likelihood ordering must assume either your choice $B_y$ or the set $\{B_y, C_y\}$.
  - On $B_y$ as well as on $\{B_y, C_y\}$, Barbara's choice $C_B$ is weakly dominated by $B_B$.
  - Thus, Barbara prefers $B_B$ to $C_B$, and $(C_B, \{B_B\})$ results as a new preference restriction for Barbara.
- Now the preference restrictions are as follows: $(A_y, \{B_y\})$, $(B_B, \{A_B\})$, and $(C_B, \{B_B\})$.
  - If you respect Barbara's preference restrictions $(B_B, \{A_B\})$ and $(C_B, \{B_B\})$, then your likelihood ordering must be $(A_B; B_B; C_B)$.
  - Hence, you assume the set $\{A_B, B_B\}$.
  - On $\{A_B, B_B\}$, your choice $B_y$ is weakly dominated by $C_y$.
  - Thus, you prefer $C_y$ to $B_y$, and $(B_y, \{C_y\})$ results as a new preference restriction for you.
- The resulting preference restrictions are: $(A_y, \{B_y\})$, $(B_y, \{C_y\})$, $(B_B, \{A_B\})$, and $(C_B, \{B_B\})$.
- Then, your only optimal choice is $C_y$.
- Indeed, $C_y$ also constitutes the only choice you can rationally and cautiously make under common full belief in (caution & respect of preferences).

# Remark

Recall that

$$u_i^k(c_i, b_i^{lex}) \leq V_i^k(r_i, b_i^{lex})$$

$$\Leftrightarrow$$

$$\sum_{c_j \in C_j} b_i^k(c_j) U_i(c_i, c_j) \leq \sum_{c_j \in C_j} b_i^k(c_j) \Big( \sum_{c_i' \in C_i} r_i(c_i') U_i(c_i', c_j) \Big)$$

$$\Leftrightarrow$$

$$\sum_{c_j \in C_j} b_i^k(c_j) U_i(c_i, c_j) \leq \sum_{c_i' \in C_i} r_i(c_i') \Big( \sum_{c_j \in C_j} b_i^k(c_j) U_i(c_i', c_j) \Big) = \sum_{c_i' \in C_i} r_i(c_i') u_i^k(c_i', b_i^{lex})$$

# Implications of Assuming a Set of Choices

## Lemma

*Suppose that player $i$ assumes a set of choices $D_j \subseteq C_j$ for opponent $j$ and let $A_i \subseteq C_i$ be some set of choices for $i$. If a choice $c_i$ is weakly dominated on $D_j$ by some randomized choice $r_i$ on $A_i$, then $i$ prefers some choice $c_i^* \in A_i$ to $c_i$.*

# Proof

- Suppose that $i$ entertains lexicographic belief $b_i^{lex} = (b_i^1; \ldots; b_i^K)$ on $C_j$, and assumes $D_j \subseteq C_j$.
- Then, $i$ deems all choices inside $D_j$ infinitely more likely than all choices outside $D_j$.
- Consequently, there exists some level $k^*$ such that
  1. for every $d_j \in D_j$ there exists $k \leq k^*$ such that $d_j \in supp(b_i^k)$,
  2. for every $c_j \in C_j \setminus D_j$ there exists no $k \leq k^*$ such that $c_j \in supp(b_i^k)$.
- Hence, the first $k^*$ levels of $b_i^{lex}$ form a cautious lexicographic belief $b_i^{lexD_j} = (b_i^1; \ldots; b_i^{k^*})$ on $D_j$.
- As $r_i$ weakly dominates $c_i$ on $D_j$, it follows that for all $k \leq k^*$ $u_i^k(c_i, b_i^{lexD_j}) = \sum_{c_j \in D_j} b_i^k(c_j) U_i(c_i, c_j) \leq \sum_{c_j \in D_j} b_i^k(c_j) V_i(r_i, c_j) = v_i^k(r_i, b_i^{lexD_j})$, and, since $b_i^{lexD_j}$ is cautious, there exists some for some $l \leq k^*$ such that $u_i^l(c_i, b_i^{lexD_j}) = \sum_{c_j \in D_j} b_i^l(c_j) U_i(c_i, c_j) \leq \sum_{c_j \in D_j} b_i^l(c_j) V_i(r_i, c_j) = v_i^l(r_i, b_i^{lexD_j})$.
- Since $u_i^k(c_i, b_i^{lexD_j}) \leq v_i^k(r_i, b_i^{lexD_j})$ for all $k \leq k^*$, it is – by Basic-Lemma II – the case for all $k \leq k^*$ that either $u_i^k(c_i, b_i^{lexD_j}) = u_i^k(a_i, b_i^{lexD_j})$ for all $a_i \in A_i$ or there exists $\hat{a}_i \in A_i$ such that $u_i^k(c_i, b_i^{lexD_j}) < u_i^k(\hat{a}_i, b_i^{lexD_j})$.
- Moreover, as $u_i^l(c_i, b_i^{lexD_j}) < v_i^l(r_i, b_i^{lexD_j})$ for some $l \leq k^*$, there must be some $l^* \leq k^*$ and – by Basic-Lemma I – some $a_i^* \in A_i$ such that $u_i^{l^*}(c_i, b_i^{lexD_j}) < u_i^{l^*}(a_i^*, b_i^{lexD_j})$, and denote the smallest such level by $l^{min}$.
- As $u_i^k(c_i, b_i^{lexD_j}) = u_i^k(a_i^*, b_i^{lexD_j})$ for all $k < l^{min}$ and $u_i^{l^{min}}(c_i, b_i^{lexD_j}) < u_i^{l^{min}}(a_i^*, b_i^{lexD_j})$, player $i$ prefers choice $a_i^*$ to $c_i$, which concludes the proof.

# Agenda

- Respecting the Opponent's Preferences
- Common Full Belief in (Caution & Respect of Preferences)
- Existence
- Towards an Algorithm
- **Algorithm**

# Example: Runaway Bride

## Story

- *You* are attending *Barbara's* wedding.
- However, when *Barbara* was supposed to say ”yes“, she suddenly changed her mind and ran away with light speed.
- *You* would like to find her and know that she is hiding in one of the following houses:

$$a \quad \rightleftharpoons \quad b \quad \rightleftharpoons \quad c \quad \rightleftharpoons \quad d \quad \rightleftharpoons \quad e$$

- *Barbara's* mother and grandmother live at $a$ and $e$, respectively, and will definitely not open the door.
- *Your* utility is $1$ if you find her, and $0$ otherwise.
- *Barbara's* utility equals simply the distance away from you.

# Example: Runaway Bride

*Barbara*

| You | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | $0, 0$ | $0, 1$ | $0, 2$ | $0, 3$ | $0, 4$ |
| $b$ | $0, 1$ | $1, 0$ | $0, 1$ | $0, 2$ | $0, 3$ |
| $c$ | $0, 2$ | $0, 1$ | $1, 0$ | $0, 1$ | $0, 2$ |
| $d$ | $0, 3$ | $0, 2$ | $0, 1$ | $1, 0$ | $0, 1$ |
| $e$ | $0, 4$ | $0, 3$ | $0, 2$ | $0, 1$ | $0, 0$ |

# Example: Runaway Bride

| You \ Barbara | $a_B$ | $b_B$ | $c_B$ | $d_B$ | $e_B$ |
|---|---|---|---|---|---|
| $a_Y$ | 0, 0 | 0, 1 | 0, 2 | 0, 3 | 0, 4 |
| $b_Y$ | 0, 1 | 1, 0 | 0, 1 | 0, 2 | 0, 3 |
| $c_Y$ | 0, 2 | 0, 1 | 1, 0 | 0, 1 | 0, 2 |
| $d_Y$ | 0, 3 | 0, 2 | 0, 1 | 1, 0 | 0, 1 |
| $e_Y$ | 0, 4 | 0, 3 | 0, 2 | 0, 1 | 0, 0 |

- What locations can you rationally and cautiously choose under common full belief in (caution & respect of preferences)?
- Observe that $c_B$ is weakly dominated by $\frac{1}{2}b_B + \frac{1}{2}d_B$ on $C_Y$.
- Thus, Barbara prefers some choice from $\{b_B, d_B\}$ to $c_B$ by the Lemma, and the preference restriction $(c_B, \{b_B, d_B\})$ for Barbara results.
- Preference restrictions: $(c_B, \{b_B, d_B\})$
  - If you respect Barbara's preference restriction $(c_B, \{b_B, d_B\})$, then you must deem either $b_B$ or $d_B$ infinitely more likely than $c_B$.
  - Hence, you will assume some set $D_B \subseteq C_B$ which includes $b_B$ or $d_B$ but not $c_B$.
  - On every such set $D_B$, your choice $c_Y$ is weakly dominated by $\frac{1}{2}b_Y + \frac{1}{2}d_Y$.
  - Thus, you prefer some choice from $\{b_Y, d_Y\}$ to $c_Y$ by the Lemma, and the preference restriction $(c_Y, \{b_Y, d_Y\})$ for you results.
  - Also, $a_Y$ and $e_Y$ are weakly dominated by $c_Y$ on $C_B$ yielding additional preference restrictions $(a_Y, \{c_Y\})$ and $(e_Y, \{c_Y\})$.

# Example: Runaway Bride

| | | Barbara | | | | |
|---|---|---|---|---|---|---|
| | | $a_B$ | $b_B$ | $c_B$ | $d_B$ | $e_B$ |
| | $a_Y$ | 0, 0 | 0, 1 | 0, 2 | 0, 3 | 0, 4 |
| | $b_Y$ | 0, 1 | 1, 0 | 0, 1 | 0, 2 | 0, 3 |
| You | $c_Y$ | 0, 2 | 0, 1 | 1, 0 | 0, 1 | 0, 2 |
| | $d_Y$ | 0, 3 | 0, 2 | 0, 1 | 1, 0 | 0, 1 |
| | $e_Y$ | 0, 4 | 0, 3 | 0, 2 | 0, 1 | 0, 0 |

- Preference restrictions: $(c_Y, \{b_Y, d_Y\})$, $(a_Y, \{c_Y\})$, $(e_Y, \{c_Y\})$, and $(c_B, \{b_B, d_B\})$
    - Note that $b_B$ and $d_B$ are weakly dominated by $\frac{3}{4}a_B + \frac{1}{4}e_B$ and $\frac{1}{4}a_B + \frac{3}{4}e_B$, respectively, on $C_Y$, yielding preference restrictions $(b_B, \{a_B, e_B\})$ and $(d_B, \{a_B, e_B\})$ for Barbara.
- Preference restrictions: $(c_Y, \{b_Y, d_Y\})$, $(a_Y, \{c_Y\})$, $(e_Y, \{c_Y\})$, as well as $(c_B, \{b_B, d_B\})$, $(b_B, \{a_B, e_B\})$, and $(d_B, \{a_B, e_B\})$.
- Therefore, only $b_Y$ and $d_Y$ can possibly be optimal for you, and only $a_B$ and $e_B$ can possibly be optimal for Barbara.

# Example: Runaway Bride

| | | Barbara | | | | |
|---|---|---|---|---|---|---|
| | | $a_B$ | $b_B$ | $c_B$ | $d_B$ | $e_B$ |
| | $a_Y$ | $0, 0$ | $0, 1$ | $0, 2$ | $0, 3$ | $0, 4$ |
| | $b_Y$ | $0, 1$ | $1, 0$ | $0, 1$ | $0, 2$ | $0, 3$ |
| You | $c_Y$ | $0, 2$ | $0, 1$ | $1, 0$ | $0, 1$ | $0, 2$ |
| | $d_Y$ | $0, 3$ | $0, 2$ | $0, 1$ | $1, 0$ | $0, 1$ |
| | $e_Y$ | $0, 4$ | $0, 3$ | $0, 2$ | $0, 1$ | $0, 0$ |

- Preference restrictions: $(c_Y, \{b_Y, d_Y\})$, $(a_Y, \{c_Y\})$, $(e_Y, \{c_Y\})$, as well as $(c_B, \{b_B, d_B\})$, $(b_B, \{a_B, e_B\})$, and $(d_B, \{a_B, e_B\})$.
- Consider the following **lexicographic epistemic model**:
  - **Type Spaces:**
    $T_{you} = \{t_y^b, t_y^d\}$ and $T_{Barbara} = \{t_B^a, t_B^e\}$
  - **Beliefs for *You*:**
    $b_{you}^{lex}(t_y^b) = ((a_B, t_B^a); (b_B, t_B^a); (e_B, t_B^a); (c_B, t_B^a); (d_B, t_B^a))$
    $b_{you}^{lex}(t_y^d) = ((e_B, t_B^e); (d_B, t_B^e); (a_B, t_B^e); (c_B, t_B^e); (b_B, t_B^e))$
  - **Beliefs for *Barbara*:**
    $b_B^{lex}(t_B^a) = ((d_Y, t_y^d); (c_Y, t_y^d); (b_Y, t_y^d); (a_Y, t_y^d); (e_Y, t_y^d))$
    $b_B^{lex}(t_B^e) = ((b_Y, t_y^b); (c_Y, t_y^b); (d_Y, t_y^b); (a_Y, t_y^b); (e_Y, t_y^b))$

# Example: Runaway Bride

| | | Barbara | | | | |
|---|---|---|---|---|---|---|
| | | $a_B$ | $b_B$ | $c_B$ | $d_B$ | $e_B$ |
| | $a_Y$ | 0, 0 | 0, 1 | 0, 2 | 0, 3 | 0, 4 |
| | $b_Y$ | 0, 1 | 1, 0 | 0, 1 | 0, 2 | 0, 3 |
| You | $c_Y$ | 0, 2 | 0, 1 | 1, 0 | 0, 1 | 0, 2 |
| | $d_Y$ | 0, 3 | 0, 2 | 0, 1 | 1, 0 | 0, 1 |
| | $e_Y$ | 0, 4 | 0, 3 | 0, 2 | 0, 1 | 0, 0 |

- All four types are cautious and express common full belief in (caution & respect of preferences).
- As $b_Y$ is optimal for $t_y^b$ and $d_Y$ is optimal for $t_y^d$, you can rationally as well as cautiously choose house $b$ and $d$ under common full belief in (caution & respect of preferences)!

# An Algorithm

**Basic Idea:** iteratively add preference restrictions to the game!

## Perea-Procedure

- **Round 1.** For every player $i$, add a preference restriction $(c_i, A_i)$, if in the full game $c_i$ is weakly dominated by some randomized choice on $A_i$.
- **Round 2.** For every player $i$, restrict to likelihood orderings $L_i$ that respect all preference restrictions for the opponent in round 1. If every such likelihood ordering $L_i$ assumes a set of opponent choices $D_j$ on which $c_i$ is weakly dominated by some randomized choice on $A_i$, then add a preference restriction $(c_i, A_i)$ for player $i$, .
- etc, until no further preference restrictions can be added.

The choices that survive this algorithm are the ones that are not part of any preference restriction generated during the complete algorithm.

**Note:** The order and speed in which preference restrictions are added is not relevant for the choices it returns.

# Algorithmic Characterization

**Theorem**

*For all $k \geq 1$, the choices that can rationally be made by a cautious type that expresses up to $k$-fold full belief in caution and respect of preferences are exactly those choices that survive the first $k+1$ steps of the Perea-Procedure.*

**Corollary**

*The choices that can rationally be made by a cautious type that expresses common full belief in (caution & respect of preferences) are exactly those choices that survive the Perea-Procedure.*

# Example: Take a Seat

## Story

- *Barbara* and *you* are the only ones to take an exam.
- Both must choose a seat.
- If both choose the same seat, then with probability $0.5$ you get the seat you want, and with probability $0.5$ you get the one horizontally next to it.
- In order to pass the exam *you* must be able copy from *Barbara*, and the same applies to her.
- A person can only copy from the other person if seated horizontally next or diagonally behind the latter.

# Example: Take a Seat

**Story** (continued)

- The probabilities of successful copying for the respective seats are given in percentages:
  $a = 0$, $b = 10$, $c = d = 20$, $e = f = 45$, $g = h = 95$
- The objective is to maximize the expected percentage of successful copying.
- Question: What seats can you rationally and cautiously choose under common full belief in (caution & respect of preferences)?

# Example: Take a Seat

Barbara

You

| | $a_B$ | $b_B$ | $c_B$ | $d_B$ | $e_B$ | $f_B$ | $g_B$ | $h_B$ |
|---|---|---|---|---|---|---|---|---|
| $a_B$ | 5, 5 | 0, 10 | 0, 0 | 0, 20 | 0, 0 | 0, 0 | 0, 0 | 0, 0 |
| $b_Y$ | 10, 0 | 5, 5 | 0, 20 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 |
| $c_Y$ | 0, 0 | 20, 0 | 20, 20 | 20, 20 | 0, 0 | 0, 45 | 0, 0 | 0, 0 |
| $d_Y$ | 20, 0 | 0, 0 | 20, 20 | 20, 20 | 0, 45 | 0, 0 | 0, 0 | 0, 0 |
| $e_Y$ | 0, 0 | 0, 0 | 0, 0 | 45, 0 | 45, 45 | 45, 45 | 0, 0 | 0, 95 |
| $f_Y$ | 0, 0 | 0, 0 | 45, 0 | 0, 0 | 45, 45 | 45, 45 | 0, 95 | 0, 0 |
| $g_Y$ | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 95, 0 | 95, 95 | 95, 95 |
| $h_Y$ | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 95, 0 | 0, 0 | 95, 95 | 95, 95 |

# Example: Take a Seat

| | | Barbara | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | $a_B$ | $b_B$ | $c_B$ | $d_B$ | $e_B$ | $f_B$ | $g_B$ | $h_B$ |
| You | $a_B$ | 5, 5 | 0, 10 | 0, 0 | 0, 20 | 0, 0 | 0, 0 | 0, 0 | 0, 0 |
| | $b_Y$ | 10, 0 | 5, 5 | 0, 20 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 |
| | $c_Y$ | 0, 0 | 20, 0 | 20, 20 | 20, 20 | 0, 0 | 0, 45 | 0, 0 | 0, 0 |
| | $d_Y$ | 20, 0 | 0, 0 | 20, 20 | 20, 20 | 0, 45 | 0, 0 | 0, 0 | 0, 0 |
| | $e_Y$ | 0, 0 | 0, 0 | 0, 0 | 45, 0 | 45, 45 | 45, 45 | 0, 0 | 0, 95 |
| | $f_Y$ | 0, 0 | 0, 0 | 45, 0 | 0, 0 | 45, 45 | 45, 45 | 0, 95 | 0, 0 |
| | $g_Y$ | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 95, 0 | 95, 95 | 95, 95 |
| | $h_Y$ | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 95, 0 | 0, 0 | 95, 95 | 95, 95 |

- **Round 1.**
  - $a_Y$ is weakly dominated by $b_Y$ on $C_B$.
  - $b_Y$ is weakly dominated by $\frac{1}{2}c_Y + \frac{1}{2}d_Y$ on $C_B$.
- With symmetry the preference restrictions $(a_Y, \{b_Y\})$ and $(b_Y, \{c_Y, d_Y\})$ as well as $(a_B, \{b_B\})$ and $(b_B, \{c_B, d_B\})$ obtain.

# Example: Take a Seat

Barbara

| You | $a_B$ | $b_B$ | $c_B$ | $d_B$ | $e_B$ | $f_B$ | $g_B$ | $h_B$ |
|---|---|---|---|---|---|---|---|---|
| $a_B$ | 5, 5 | 0, 10 | 0, 0 | 0, 20 | 0, 0 | 0, 0 | 0, 0 | 0, 0 |
| $b_Y$ | 10, 0 | 5, 5 | 0, 20 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 |
| $c_Y$ | 0, 0 | 20, 0 | 20, 20 | 20, 20 | 0, 0 | 0, 45 | 0, 0 | 0, 0 |
| $d_Y$ | 20, 0 | 0, 0 | 20, 20 | 20, 20 | 0, 45 | 0, 0 | 0, 0 | 0, 0 |
| $e_Y$ | 0, 0 | 0, 0 | 0, 0 | 45, 0 | 45, 45 | 45, 45 | 0, 0 | 0, 95 |
| $f_Y$ | 0, 0 | 0, 0 | 45, 0 | 0, 0 | 45, 45 | 45, 45 | 0, 95 | 0, 0 |
| $g_Y$ | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 95, 0 | 95, 95 | 95, 95 |
| $h_Y$ | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 95, 0 | 0, 0 | 95, 95 | 95, 95 |

- **Round 2.** preference restrictions: $(a_Y, \{b_Y\}), (b_Y, \{c_Y, d_Y\}), (a_B, \{b_B\}), (b_B, \{c_B, d_B\})$
  - If you respect preference restriction $(a_B, \{b_B\})$, then you must assume some set $D_B \subseteq C_B$ which contains $b_B$ but not $a_B$.
  - For every such set $D_B$ it holds that $d_Y$ is weakly dominated by $c_Y$.
  - Moreover, if you respect preference restrictions $(a_B, \{b_B\})$ and $(b_B, \{c_B, d_B\})$, then you must assume some set $D_B \subseteq C_B$ which contains $c_B$ or $d_B$ but not $a_B$ and not $b_B$ .
  - For every such set $D_B$ it holds that $c_Y$ is weakly dominated by $\frac{1}{2}e_Y + \frac{1}{2}f_Y$.

# Example: Take a Seat

| | | Barbara | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | $a_B$ | $b_B$ | $c_B$ | $d_B$ | $e_B$ | $f_B$ | $g_B$ | $h_B$ |
| You | $a_B$ | 5, 5 | 0, 10 | 0, 0 | 0, 20 | 0, 0 | 0, 0 | 0, 0 | 0, 0 |
| | $b_Y$ | 10, 0 | 5, 5 | 0, 20 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 |
| | $c_Y$ | 0, 0 | 20, 0 | 20, 20 | 20, 20 | 0, 0 | 0, 45 | 0, 0 | 0, 0 |
| | $d_Y$ | 20, 0 | 0, 0 | 20, 20 | 20, 20 | 0, 45 | 0, 0 | 0, 0 | 0, 0 |
| | $e_Y$ | 0, 0 | 0, 0 | 0, 0 | 45, 0 | 45, 45 | 45, 45 | 0, 0 | 0, 95 |
| | $f_Y$ | 0, 0 | 0, 0 | 45, 0 | 0, 0 | 45, 45 | 45, 45 | 0, 95 | 0, 0 |
| | $g_Y$ | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 95, 0 | 95, 95 | 95, 95 |
| | $h_Y$ | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 95, 0 | 0, 0 | 95, 95 | 95, 95 |

- **Round 3.** preference restrictions: $(a_Y, \{b_Y\})$, $(b_Y, \{c_Y, d_Y\})$, $(d_Y, \{c_Y\})$, $(c_Y, \{e_Y, f_Y\})$, $(a_B, \{b_B\})$, $(b_B, \{c_B, d_B\})$, $(d_B, \{c_B\})$, $(c_B, \{e_B, f_B\})$
  - If you respect preference restriction $(d_B, \{c_B\})$, then you must assume some set $D_B \subseteq C_B$ which contains $c_B$ but not $d_B$.
  - For every such set $D_B$ it holds that $e_Y$ is weakly dominated by $f_Y$.
  - Moreover, if you respect preference restrictions $(a_B, \{b_B\})$, $(b_B, \{c_B, d_B\})$, $(d_B, \{c_B\})$, $(c_B, \{e_B, f_B\})$, then you must assume some set $D_B \subseteq C_B$ which contains $e_B$ or $f_B$ but not any choice from $\{a_B, b_B, c_B, d_B\}$ .
  - For every such set $D_B$ it holds that $f_Y$ is weakly dominated by $\frac{1}{2} g_Y + \frac{1}{2} h_Y$.

# Example: Take a Seat

Barbara

You

| | $a_B$ | $b_B$ | $c_B$ | $d_B$ | $e_B$ | $f_B$ | $g_B$ | $h_B$ |
|---|---|---|---|---|---|---|---|---|
| $a_B$ | 5, 5 | 0, 10 | 0, 0 | 0, 20 | 0, 0 | 0, 0 | 0, 0 | 0, 0 |
| $b_Y$ | 10, 0 | 5, 5 | 0, 20 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 |
| $c_Y$ | 0, 0 | 20, 0 | 20, 20 | 20, 20 | 0, 0 | 0, 45 | 0, 0 | 0, 0 |
| $d_Y$ | 20, 0 | 0, 0 | 20, 20 | 20, 20 | 0, 45 | 0, 0 | 0, 0 | 0, 0 |
| $e_Y$ | 0, 0 | 0, 0 | 0, 0 | 45, 0 | 45, 45 | 45, 45 | 0, 0 | 0, 95 |
| $f_Y$ | 0, 0 | 0, 0 | 45, 0 | 0, 0 | 45, 45 | 45, 45 | 0, 95 | 0, 0 |
| $g_Y$ | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 95, 0 | 95, 95 | 95, 95 |
| $h_Y$ | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 95, 0 | 0, 0 | 95, 95 | 95, 95 |

- **Round 4.** preference restrictions: $(a_Y, \{b_Y\})$, $(b_Y, \{c_Y, d_Y\})$, $(d_Y, \{c_Y\})$, $(c_Y, \{e_Y, f_Y\})$, $(e_Y, \{f_Y\})$, $(f_Y, \{g_Y, h_Y\})$, $(a_B, \{b_B\})$, $(b_B, \{c_B, d_B\})$, $(d_B, \{c_B\})$, $(c_B, \{e_B, f_B\})$, $(e_B, \{f_B\})$, $(f_B, \{g_B, h_B\})$
  - If you respect preference restriction $(e_B, \{f_B\})$, then you must assume some set $D_B \subseteq C_B$ which contains $f_B$ but not $e_B$.
  - For every such set $D_B$ it holds that $h_Y$ is weakly dominated by $g_Y$.
  - However, note that with preference restrictions $(a_Y, \{b_Y\})$, $(b_Y, \{c_Y, d_Y\})$, $(d_Y, \{c_Y\})$, $(c_Y, \{e_Y, f_Y\})$, $(e_Y, \{f_Y\})$, $(f_Y, \{g_Y, h_Y\})$, $(h_Y, \{g_Y\})$, only your choice $g_Y$ can be optimal!
- Under common full belief in (caution & respect of preferences), you can thus only rationally and cautiously take seat $g$.

# Example: Take a Seat

| | | Barbara | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | $a_B$ | $b_B$ | $c_B$ | $d_B$ | $e_B$ | $f_B$ | $g_B$ | $h_B$ |
| You | $a_B$ | 5, 5 | 0, 10 | 0, 0 | 0, 20 | 0, 0 | 0, 0 | 0, 0 | 0, 0 |
| | $b_Y$ | 10, 0 | 5, 5 | 0, 20 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 |
| | $c_Y$ | 0, 0 | 20, 0 | 20, 20 | 20, 20 | 0, 0 | 0, 45 | 0, 0 | 0, 0 |
| | $d_Y$ | 20, 0 | 0, 0 | 20, 20 | 20, 20 | 0, 45 | 0, 0 | 0, 0 | 0, 0 |
| | $e_Y$ | 0, 0 | 0, 0 | 0, 0 | 45, 0 | 45, 45 | 45, 45 | 0, 0 | 0, 95 |
| | $f_Y$ | 0, 0 | 0, 0 | 45, 0 | 0, 0 | 45, 45 | 45, 45 | 0, 95 | 0, 0 |
| | $g_Y$ | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 95, 0 | 95, 95 | 95, 95 |
| | $h_Y$ | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 95, 0 | 0, 0 | 95, 95 | 95, 95 |

- Consider the following **lexicographic epistemic model**:
  - **Type Spaces:** $T_{you} = \{t_Y\}$ and $T_{Barbara} = \{t_B\}$
  - **Beliefs for *You*:** $b_{you}(t_Y) = ((g_B, t_B); (h_B, t_B); (f_B, t_B); (e_B, t_B); (c_B, t_B); (d_B, t_B); (b_B, t_B); (a_B, t_B))$
  - **Beliefs for *Barbara*:** $b_B(t_B) = ((g_Y, t_Y); (h_Y, t_Y); (f_Y, t_Y); (e_Y, t_Y); (c_Y, t_Y); (d_Y, t_Y); (b_Y, t_Y); (a_Y, t_Y))$

**Thank you!**