Incomplete information games

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Common knowledge of the game structure

- So far we have assumed that *the structure of the game is common knowledge*.
- Recall that the game structure is described by (C_a, C_b, u_a, u_b).
- Thus, CK of the game structure means that

  (K_1) every player knows that the game (C_a, C_b, u_a, u_b) is played.
  (K_2) every player knows that (K_1) holds.
  (K_3) every player knows that (K_2) holds.
  $\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

- Now, we will partially relax this assumption. We will study games where
  - the choice sets (C_a, C_b) are CK
  - the preferences/utility functions (u_a, u_b) are not CK

Incomplete information vs. strategic uncertainty

- There are two types of uncertainty:
  - **Incomplete information:** uncertainty about the preferences
  - **Strategic uncertainty/imperfect information:** uncertainty about the opponent's choices
- Strategic uncertainty always described by subjective beliefs
- Incomplete information may be described either by subjective or by objective beliefs (depending on the source of uncertainty).

An auction example

Example (First-price auction)

Let Ann and Bob be the two bidders in a first-price auction for a bike. Each of the bidders has a *private value*, which is the highest willingness to pay. Then, Ann's utility function depends on her private value, v_a:

$$u_a(c_a, c_b | v_a) = \begin{cases} v_a - c_a & \text{if } c_a > c_b, \\ (v_a - c_a)/2 & \text{if } c_a = c_b, \\ 0 & \text{if} c_a < c_b. \end{cases}$$

Crucial difference: Her private value, and therefore her utility function *may or may not be known to Bob*.

Ann's utility functions in the auction

- Suppose that Bob is uncertain about Ann's private value.
- Still, somehow he knows that her private value is either 2 or 4.
- This means that Bob does not know if Ann's utility function is

$$\boxed{u_a(c_a, c_b|v_a = 2) = \begin{cases} 2 - c_a & \text{if } c_a > c_b, \\ (2 - c_a)/2 & \text{if } c_a = c_b, \\ 0 & \text{if} c_a < c_b. \end{cases}} \text{ OR } \boxed{u_a(c_a, c_b|v_a = 4) = \begin{cases} 4 - c_a & \text{if } c_a > c_b, \\ (4 - c_a)/2 & \text{if } c_a = c_b, \\ 0 & \text{if} c_a < c_b. \end{cases}}$$

- Thus, Bob is uncertain about which utility function from

$$\mathcal{U}_a = \{u_a(\cdot|v_a) \mid v_a \in \mathbb{R}\}$$

  is the actual one.
- Each of them corresponds to a different game.

Modelling incomplete information games

- We begin with the set of utility functions $\mathcal{U} = \mathcal{U}_a \times \mathcal{U}_b$.
- The players (may) have uncertainty about $\mathcal{U}$.
- We introduce Bob's **belief hierarchy about the utility functions:**
  - Bob's belief about $\mathcal{U}$
  - Bob's belief about Ann's beliefs about $\mathcal{U}$
  - Bob's belief about Ann's beliefs about his beliefs about $\mathcal{U}$
- We can model interactive uncertainty about $\mathcal{U}$.
- Uncertainty can be either objective or subjective. What does this mean? Can we think of examples.
- We allow for uncertainty about his own utility function. How?
- Henceforth we will mostly focus on cases where players know their own utility function.

Harsanyi's model of incomplete information games

- Bob's belief hierarchies can be modelled with a type space:
  - A (finite) set of types Θ_b
  - A belief function $\lambda_b : \Theta_b \to \Delta(\mathcal{U} \times \Theta_a)$
- These types represent belief hierarchies about utility functions, *not about choices*.
- So they differ conceptually from belief types.

Harsanyi's model of incomplete information games

- Each Θ_a-type can be thought as a different self of the same player (different "player").
- So we obtain a new larger game, where the set of players is not $\{a, b\}$ anymore, but rather $\Theta_a \cup \Theta_b$.
- For each player $\theta_b \in \Theta_b$ we have
  - a set of relevant opponents from Θ_a. Which ones?
  - a utility function $u_b(\cdot|\theta_b)$.
- **Remark:** Bob knowing his own utility function does not mean that $u_b(\cdot|\theta_b)$ is the same for every θ_b. It rather means that

$$\lambda_b(\theta_b)\Big(\mathcal{U}_a \times \{u_b(\cdot|\theta_b)\} \times \Theta_a\Big) = 1.$$

Auction example

- Ann knows her value ($v_a = 2$ or $v_a = 4$) and Bob's ($v_b = 3$).
- Bob knows his own value, but not Ann's (equal prob).
- We take the types $\Theta_a = \{\theta_a^2, \theta_a^4\}$ and $\Theta_b = \{\theta_b^3\}$ with

$$\begin{aligned}
\lambda_a(\theta_a^2)\big(\{u_a^2, u_b^3\} \times \{\theta_b^3\}\big) &= 1 \\
\lambda_a(\theta_a^4)\big(\{u_a^4, u_b^3\} \times \{\theta_b^3\}\big) &= 1 \\
\lambda_b(\theta_b^3)\big(\{u_a^2, u_b^3\} \times \{\theta_a^2\}\big) &= \lambda_b(\theta_b^3)\big(\{u_a^4, u_b^3\} \times \{\theta_a^4\}\big) = 1
\end{aligned}$$

  where $u_i^k := u_i(\cdot|\theta_i^k)$.
- What if Ann did not know Bob's private value?

Auction example

- What is the functional form of u_b^3?
- Bob does not know if he plays against θ_a^2 or θ_a^4.
- The choices of the two types need not be the same.
- Hence, $u_b^3 : C_a \times C_a \times C_b \to \mathbb{R}$ is given by

$$u_b(c_a^2, c_a^4, c_b|\theta_b^3) = \begin{cases} 3 - c_b & \text{if } c_b > c_a^2 \text{ and } c_b > c_a^4, \\ \frac{3}{4}(3 - c_b) & \text{if } c_b > c_a^2 \text{ and } c_b = c_a^4, \\ \frac{1}{2}(3 - c_b) & \text{if } c_b > c_a^2 \text{ and } c_b < c_a^4, \\ \frac{3}{4}(3 - c_b) & \text{if } c_b = c_a^2 \text{ and } c_b > c_a^4, \\ \frac{1}{2}(3 - c_b) & \text{if } c_b = c_a^2 \text{ and } c_b = c_a^4, \\ \frac{1}{4}(3 - c_b) & \text{if } c_b = c_a^2 \text{ and } c_b < c_a^4, \\ \frac{1}{2}(3 - c_b) & \text{if } c_b < c_a^2 \text{ and } c_b > c_a^4, \\ \frac{1}{4}(3 - c_b) & \text{if } c_b < c_a^2 \text{ and } c_b = c_a^4, \\ 0 & \text{if } c_b < c_a^2 \text{ and } c_b < c_a^4 \end{cases}$$

How did we get this?

- Uncertainty due to incomplete information.
- Draw the incomplete information game if $C_i = \{1, 2, 3, 4\}$.

Complete information games

- Complete information means that (u_a, u_b) are CK.
- This is a special case within Harsanyi's framework.
- Each player has only one type which is certain about its utility function.
  - $\Theta_a = \{\theta_a\}$ and $\Theta_b = \{\theta_b\}$
  - $\lambda_i(\theta_i)\big(\{u_i, u_j\} \times \{\theta_j\}\big) = 1$
- How does our auction example become when it is CK that $v_a = 3$ and $v_b = 2$?

Solution concepts for incomplete information games

- We will look at some standard solution concepts for games with incomplete information.
- **Interim correlated rationalizability:** the analogue of RCBR.
- **Bayesian equilibrium:** the analogue of NE.
- The underlying idea is quite simple: roughly, we apply the complete-information counterparts to the extended game.
- To do this, we first need to augment the incomplete information game with beliefs that describe strategic uncertainty (on top of the uncertainty about the utility functions).

Epistemic model

- Bob does not know if he plays against θ_a^2 or θ_a^4.
- The choices of the two types need not be the same.
- So he forms beliefs about the choices of Ann separately for θ_a^2 and θ_a^4.
- In general, our epistemic model consists of:
    - a finite set of types T_{θ_i} for each $\theta_i \in \Theta_i$
    - a belief mapping $b_{\theta_i} : T_{\theta_i} \to \Delta\Big(\times_{\theta_j \in \Theta_j}(C_{\theta_j} \times T_{\theta_j})\Big)$
- Of course the only relevant θ_j's are those that receive positive probability by $\lambda_i(\theta_i)$. Why?

Expected utility

- Suppose that $T_{\theta_a^2} = \{t_a^2, \tilde{t}_a^2\}$, $T_{\theta_a^4} = \{t_a^4\}$ and $T_{\theta_b^3} = \{t_b^3\}$.
- Bob's subjective beliefs are such that

$$b_{\theta_b}(t_b^3) = \Big(\frac{1}{4}\otimes((1, t_a^2), (2, t_a^4)) \;;\; \frac{1}{4}\otimes((2, t_a^2), (3, t_a^4)) \;;\; \frac{1}{2}\otimes((2, \tilde{t}_a^2), (4, t_a^4))\Big)$$

How should we read the previous expression?

- Then, t_b^3's expected utility from choosing $c_b = 2$ becomes

$$U_b(2, t_b^3) = \frac{1}{4}\frac{3}{4}(3-2) + \frac{1}{4}\frac{1}{4}(3-2) + \frac{1}{2}\frac{1}{2}(3-2) = \frac{1}{2}$$

How did we get this?

- **Remark:** This expected utility has two sources of uncertainty. Which ones?

Expected utility

- Formally, take some $t_i \in T_{\theta_i}$.
- For simplicity,
  - we enumerate the Θ_j-types, i.e., $\Theta_j = \{\theta_j^1, \ldots, \theta_j^n\}$
  - we write $C_j^k := C_{\theta_j^k} = C_j$ and $C_j := C_j^1 \times \cdots \times C_j^n$
- Then, the expected utility (from the choice c_i) is given by

$$U_i(c_i, t_i) = \sum_{(c_j^1, \ldots, c_j^n) \in C_j} b_{\theta_i}(t_i)(c_j^1, \ldots, c_j^n) u_i(c_i, c_j^1, \ldots, c_j^n | \theta_i)$$

- **Remark:** $u_i(c_i, c_j^1, \ldots, c_j^n | \theta_i)$ is also an expected utility that incorporates the uncertainty from the incomplete information.

Rationality and solution concepts

- Rationality is defined in the usual way.
- A choice-type pair $(c_i, t_i) \in C_i \times T_{\theta_i}$ is **rational** if $U_i(c_i, t_i) \geq U_i(c_i', t_i)$ for every $c_i' \in C_i$.
- Similarly to complete-information games, a **solution concept** receives the game as an input and returns a set of choice profiles as an output.
- The only difference is that here the predicted choice profiles are not elements of $C_a \times C_b$ but rather of

$$\Big(\bigtimes_{\theta_a \in \Theta_a} C_{\theta_a}\Big) \times \Big(\bigtimes_{\theta_b \in \Theta_b} C_{\theta_b}\Big)$$

Why is this the case?

Interim correlated rationalizability

- In complete information games correlated rationalizability yields the choice profiles that survive IESDC.
- The only difference here is that we apply IESDC in the extended game.
- That is, we treat each Θ_i-type as a different player.
- The choice profiles that we obtain are called **interim correlated rationalizable** (Dekel, Fudenberg & Morris, 2007).
- **Remark:** If the game is one with complete information, ICR collapses to RCBR.

Bayesian equilibrium

- In complete information games NE yields the (mixed) choice profiles that are best response to each other.
- The only difference here is that we apply NE in the extended game.
- That is, again we treat each Θ_i-type as a different player.
- The choice profiles that we obtain are called **Bayesian equilibria** (Harsanyi, 1967-68).
- **Remark:** In Harsanyi's original formulation there is a common prior. However, the idea remains the same.

Questions???