EpiCenter Spring Course on Epistemic Game Theory
Chapter 4: Simple Belief Hierarchies

Andrés Perea

Maastricht University

July 2016

Simple belief hierarchies

- Previously, we have discussed the idea of common belief in rationality.
- So, we focus on belief hierarchies in which you believe that
- your opponents choose rationally,
- your opponents believe that their opponents choose rationally,
- your opponents believe that their opponents believe that their opponents choose rationally,
- and so on.
- Can we still distinguish between such belief hierarchies?
- We will look at psychological factors beyond common belief in rationality.

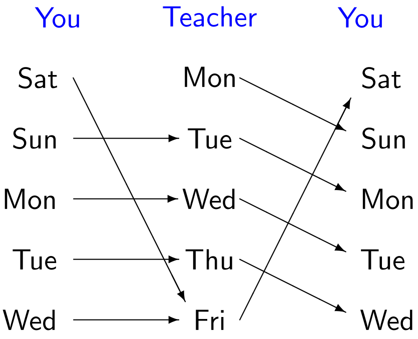
Example: Teaching a lesson

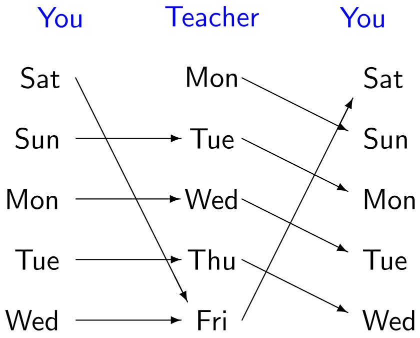
Story

- It is Friday, and your biology teacher tells you that he will give you a surprise exam next week.
- You must decide on what day you will start preparing for the exam.
- In order to pass the exam, you must study for at least two days.
- To write the perfect exam, you must study for at least six days. In that case, you will get a compliment by your father.
- Passing the exam increases your utility by 5.
- Failing the exam increases the teacher's utility by 5.
- Every day you study decreases your utility by 1, but increases the teacher's utility by 1.
- A compliment by your father increases your utility by 4.

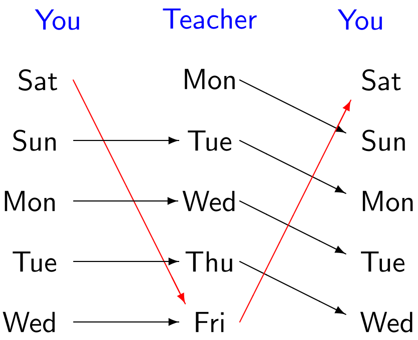
Teacher

You		Mon	Tue	Wed	Thu	Fri
	Sat	3, 2	2, 3	1, 4	0, 5	3, 6
	Sun	−1, 6	3, 2	2, 3	1, 4	0, 5
	Mon	0, 5	−1, 6	3, 2	2, 3	1, 4
	Tue	0, 5	0, 5	−1, 6	3, 2	2, 3
	Wed	0, 5	0, 5	0, 5	−1, 6	3, 2

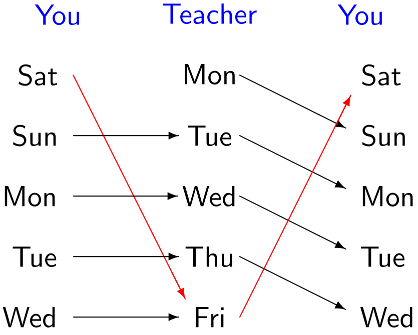




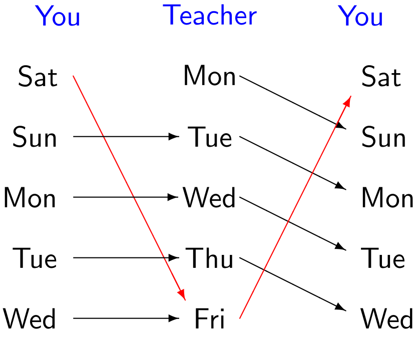
- Under common belief in rationality, you can rationally choose any day to start studying.
- Is there still a way to distinguish between your various choices?
- Yes! Some choices are supported by a simple belief hierarchy, whereas other choices are not.



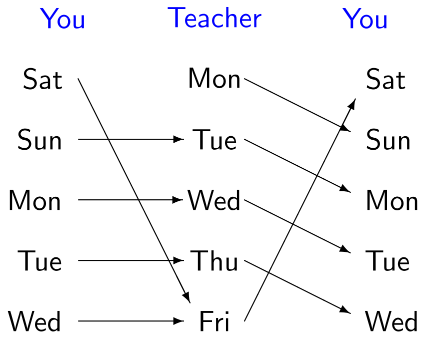
- Consider the belief hierarchy that supports your choices Saturday and Wednesday.
- This belief hierarchy is entirely generated by the belief σ_2 that the teacher puts the exam on Friday, and the belief σ_1 that you start studying on Saturday.



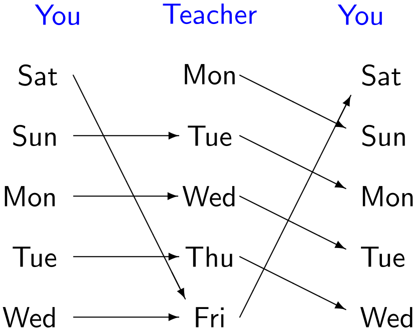
- Let σ_2 be the belief that the teacher chooses *Friday,* and let σ_1 be the belief that you choose *Saturday.*
- Then, in the belief hierarchy that supports your choices *Saturday* and *Wednesday,*
- your belief about the teacher's choice is σ_2,
- you believe, with probability 1, that the teacher's belief about your choice is σ_1,
- ...



- ... you believe, with prob. 1, that the teacher believes, with prob. 1, that your belief about the teacher's choice is indeed σ_2,
- you believe, with prob. 1, that the teacher believes, with prob. 1, that you believe, with prob. 1,that the teacher's belief about your choice is indeed σ_1,
- and so on.
- So, this belief hierarchy is completely generated by the beliefs σ_1 and σ_2. We call such a belief hierarchy simple.



- The belief hierarchies that support your choices Sunday, ... ,Tuesday are certainly not simple. Consider, for instance, the belief hierarchy that supports your choice Sunday. There,
- you believe that the teacher puts the exam on Tuesday,
- but you believe that the teacher believes that you believe that the teacher will put the exam on Wednesday.



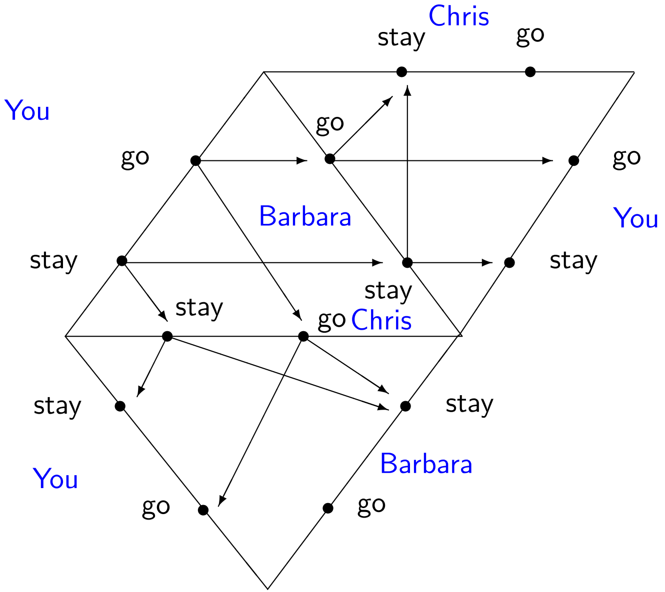
Summarizing

- Within this beliefs diagram:
- You can rationally make every choice under common belief in rationality.
- Your choices Saturday and Wednesday are supported by a simple belief hierarchy.
- Your other choices are supported by non-simple belief hierarchies.

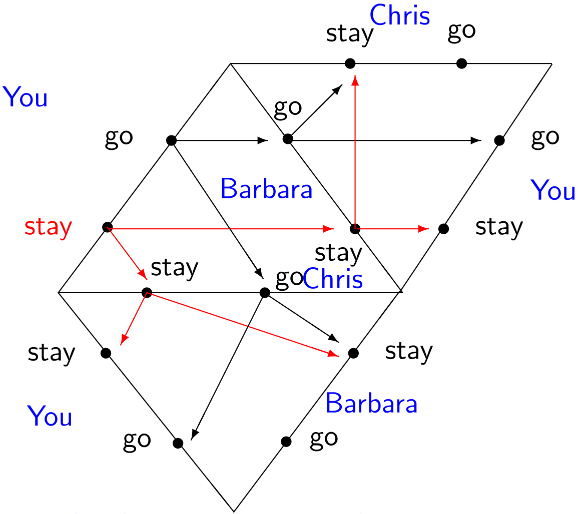
Example: Movie or party?

Story

- You have been invited to a party this evening, together with Barbara and Chris. But this evening, your favorite movie Once upon a time in America, starring Robert de Niro, will be on TV.
- Having a good time at the party gives you utility 3, watching the movie gives you utility 2, whereas having a bad time at the party gives you utility 0. Similarly for Barbara and Chris.
- You will only have a good time at the party if Barbara and Chris both join.
- Barbara and Chris had a fierce discussion yesterday. Barbara will only have a good time at the party if you join, but not Chris.
- Chris will only have a good time at the party if you join, but not Barbara.
- What should you do: Go to the party, or stay at home?

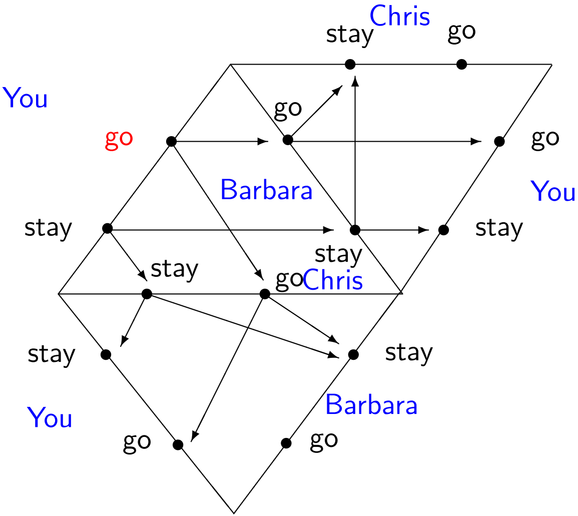


- Under common belief in rationality, you can go to the party or stay at home.

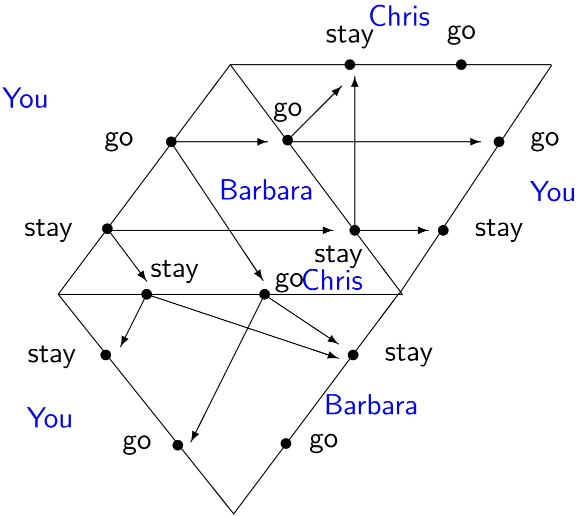


- The belief hierarchy that supports your choice stay is simple: It is completely generated by the beliefs

$$\sigma_1 = \text{ You stay, } \sigma_2 = \text{ Barbara stays, } \sigma_3 = \text{ Chris stays.}$$



- The belief hierarchy that supports your choice go is not simple:
- You believe that Chris will go to the party.
- You believe that Barbara believes that Chris will stay at home.



- Summarizing: Under common belief in rationality, you can rationally choose go or stay.
- In this beliefs diagram, stay is supported by a simple belief hierarchy, but go is not.

- In general, a belief hierarchy is called simple if it is generated by a combination of beliefs $\sigma_1, ..., \sigma_n$.

Definition (Belief hierarchy generated by $(\sigma_1, ..., \sigma_n)$)

For every player i, let σ_i be a probabilistic belief about i's choice.

The belief hierarchy for player i that is generated by $(\sigma_1, ..., \sigma_n)$ states that

(1) player i has belief σ_j about player j's choice,

(2) player i believes that player j has belief σ_k about player k's choice,

(3) player i believes that player j believes that player k has belief σ_l about player l's choice,

and so on.

Definition (Simple belief hierarchy)

Consider an epistemic model, and a type t_i within it.

Type t_i has a simple belief hierarchy, if its belief hierarchy is generated by some combination of beliefs $(\sigma_1, ..., \sigma_n)$.

- Observation 1: A type with a simple belief hierarchy always believes that his opponents are correct about his entire belief hierarchy.
- Proof. Take a type t_i with a simple belief hierarchy. Then, its belief hierarchy is generated by some combination of beliefs $(\sigma_1, ..., \sigma_n)$.
- Fix an opponent j. Then, t_i has belief σ_j about j's choice. But also, t_i believes that every opponent believes that he (player i) has indeed belief σ_j about j's choice.
- Fix an opponent j, and some player $k \neq j$. Then, t_i believes that player j has belief σ_k about k's choice. But also, t_i believes that every opponent believes that he (player i) indeed believes that player j has belief σ_k about k's choice.
- And so on.

Definition (Simple belief hierarchy)

Consider an epistemic model, and a type t_i within it.

Type t_i has a simple belief hierarchy, if its belief hierarchy is generated by some combination of beliefs $(\sigma_1, ..., \sigma_n)$.

- Observation 2: In a game with three players or more, a type t_i with a simple belief hierarchy believes that his opponents share his beliefs about other players.

- Proof. Suppose that t_i's belief hierarchy is generated by $(\sigma_1, ..., \sigma_n)$.
- Fix two different opponents j and k. Then, t_i's belief about k's choice is σ_k. But t_i also believes that j has belief σ_k about k's choice.
- Take some player $l \neq k$. Then, t_i believes that k's belief about l's choice is σ_l. But t_i also believes that j believes that k's belief about l's choice is σ_l.
- And so on. ■

Nash equilibrium

- Previously we have focused on belief hierarchies that express common belief in rationality.
- So far in this chapter, we have focused on belief hierarchies that are simple.
- Can we characterize, in an easy way, those belief hierarchies that express common belief in rationality and are simple?

- Consider a type t_i with a simple belief hierarchy. Then, t_i's belief hierarchy is generated by some combination $(\sigma_1, ..., \sigma_n)$ of beliefs. Hence:

- t_i's belief about the opponents' choices is σ_{-i},

- t_i believes that player j's has belief σ_{-j} about his opponents' choices,

- t_i believes that player j believes that player k has belief σ_{-k} about his opponents' choices,

- and so on.

- Suppose that, in addition, type t_i expresses common belief in rationality.

- Take some opponent's choice c_j with $\sigma_j(c_j) > 0$.
- Then, t_i assigns positive probability to c_j.
- As t_i believes in j's rationality, choice c_j must be optimal for player j under the belief σ_{-j} about the opponents' choices.

- Now, take some own choice c_i with $\sigma_i(c_i) > 0$.
- Then, type t_i believes that every opponent j assigns positive probability to c_i.
- As t_i believes that j believes in i's rationality, choice c_i must be optimal for player i under the belief σ_{-i} about the opponents' choices.

- Conclusion: If t_i is a type that

- has a simple belief hierarchy, generated by the combination of beliefs $(\sigma_1, .., \sigma_n)$, and
- expresses common belief in rationality,

- then, for every player j, the belief σ_j only assigns positive probability to choices c_j that are optimal under the belief σ_{-j}.

Definition (Nash equilibrium)

The combination of beliefs $(\sigma_1, ..., \sigma_n)$ is a Nash equilibrium if for every player j, the belief σ_j only assigns positive probability to choices c_j that are optimal under the belief σ_{-j}.

Theorem

Consider a type t_i which

(1) has a simple belief hierarchy, generated by the combination $(\sigma_1, ..., \sigma_n)$ of beliefs, and

(2) expresses common belief in rationality.

Then, the combination of beliefs $(\sigma_1, ..., \sigma_n)$ must be a Nash equilibrium.

- We can show that also the opposite direction is true.

Theorem

Consider a type t_i with a simple belief hierarchy, generated by the combination $(\sigma_1, ..., \sigma_n)$ of beliefs.

If the combination of beliefs $(\sigma_1, ..., \sigma_n)$ is a Nash equilibrium, then type t_i expresses common belief in rationality.

- Proof. We first show that t_i believes in his opponents' rationality.
- Take an opponent j, and assume that t_i assigns positive probability to choice c_j.
- Then $\sigma_j(c_j) > 0$, and hence c_j must be optimal for player j under the belief σ_{-j}.
- Since t_i believes that j's belief about the opponents' choices is σ_{-j}, type t_i believes that c_j is optimal for player j.
- So, t_i only assigns positive probability to a choice c_j if he believes that c_j is optimal for player j.
- Hence, type t_i believes in his opponents' rationality.

- Proof continued. We next show that t_i believes that his opponents believe in their opponents' rationality.

- Take an opponent j, and some player $k \neq j$. Suppose, t_i believes that player j assigns positive probability to choice c_k.
- Then $\sigma_k(c_k) > 0$, and hence c_k must be optimal for player k under the belief σ_{-k}.
- Since t_i believes that player j believes that k's belief about his opponents' choices is σ_{-k}, type t_i believes that player j believes that c_k is optimal for player k.
- So, if t_i believes that player j assigns positive probability to choice c_k, then t_i believes that player j believes that c_k is optimal for player k.
- Hence, type t_i believes that player j believes in k's rationality.
- As such, type t_i believes that his opponents believe in their opponents' rationality.
- And so on. ■

- By combining the two theorems above, we obtain the following characterization.

Theorem (Simple belief hierarchies versus Nash equilibrium)

Consider a type t_i with a simple belief hierarchy, generated by the combination $(\sigma_1, ..., \sigma_n)$ of beliefs.

Then, type t_i expresses common belief in rationality, if and only if, the combination of beliefs $(\sigma_1, ..., \sigma_n)$ is a Nash equilibrium.

- Important consequence:
- Suppose that in a given game, we wish to find the simple belief hierarchies that express common belief in rationality.
- Then, it is sufficient to find all the Nash equilibria $(\sigma_1, ..., \sigma_n)$ in the game.

- Question: Can we always find simple belief hierarchies that express common belief in rationality?
- The answer is given by John Nash, in his PhD dissertation.

Theorem (Nash equilibrium always exists)

For every game with finitely many choices there is at least one Nash equilibrium $(\sigma_1, ..., \sigma_n)$.

Theorem (Common belief in rationality with simple belief hierarchies is always possible)

Consider a game with finitely many choices. Then, for every player i there is at least one simple belief hierarchy that expresses common belief in rationality.

Computing Nash equilibria

- We wish to find those choices you can rationally make if you
- express common belief in rationality, and
- hold a simple belief hierarchy.
- Is there a method to find these choices?

- Consider a type t_i with a simple belief hierarchy, generated by the combination $(\sigma_1, ..., \sigma_n)$ of beliefs.
- Remember: Type t_i expresses common belief in rationality, if and only if, the combination $(\sigma_1, ..., \sigma_n)$ of beliefs is a Nash equilibrium.
- Moreover, choice c_i is optimal for t_i if c_i is optimal under the belief σ_{-i} about the opponents' choices.
- Hence, choice c_i can rationally be made under common belief in rationality with a simple belief hierarchy, if and only if, there is some Nash equilibrium $(\sigma_1, ..., \sigma_n)$ where c_i is optimal under σ_{-i}.

Definition (Nash choice)

A choice c_i is a Nash choice if there is some Nash equilibrium $(\sigma_1, ..., \sigma_n)$ where c_i is optimal for player i under the belief σ_{-i}.

Definition (Nash choice)

A choice c_i is a Nash choice if there is some Nash equilibrium $(\sigma_1, ..., \sigma_n)$ where c_i is optimal for player i under the belief σ_{-i}.

- Observation 1: If there is a Nash equilibrium $(\sigma_1, ..., \sigma_n)$ with $\sigma_i(c_i) > 0$, then c_i is a Nash choice.
- Proof: Take some choice c_i with $\sigma_i(c_i) > 0$. Since $(\sigma_1, ..., \sigma_n)$ is a Nash equilibrium, c_i is optimal under the belief σ_{-i}.
- Hence, c_i is a Nash choice. ■

Definition (Nash choice)

A choice c_i is a Nash choice if there is some Nash equilibrium $(\sigma_1, ..., \sigma_n)$ where c_i is optimal for player i under the belief σ_{-i}.

- Observation 2: A Nash choice c_i need not always receive positive probability in a Nash equilibrium.
- Proof: Consider the game

	c	d
a	$2, 0$	$0, 1$
b	$1, 0$	$1, 0$

.

- Then, $(b, \frac{1}{2}c + \frac{1}{2}d)$ is a Nash equilibrium.
- Since a is optimal under the belief $\frac{1}{2}c + \frac{1}{2}d$, choice a is a Nash choice.
- However, there is no Nash equilibrium (σ_1, σ_2) with $\sigma_1(a) > 0$.
- Indeed, if $\sigma_1(a) > 0$, then only d is optimal for player 2, and hence $\sigma_2 = d$.
- But then, only b can be optimal for player 1, hence $\sigma_1 = b$. This is a contradiction. ■

Theorem (Simple belief hierarchies versus Nash choices)

Player i can rationally make choice c_i under common belief in rationality with a simple belief hierarchy, if and only if, c_i is a Nash choice.

- Suppose we wish to find those choices that player i can make if
- he holds a simple belief hierarchy, and
- he expresses common belief in rationality.
- Then, it is sufficient to compute all Nash choices for player i in the game.
- Bad news: There is no simple algorithm for computing all Nash equilibria in a game.
- In some games, this is a difficult task.

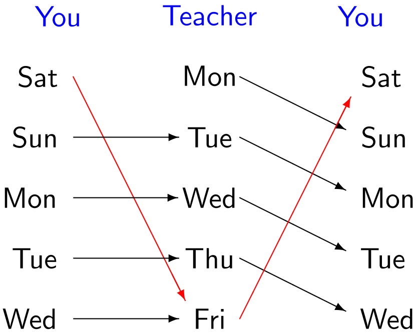
Example: Teaching a lesson

Teacher

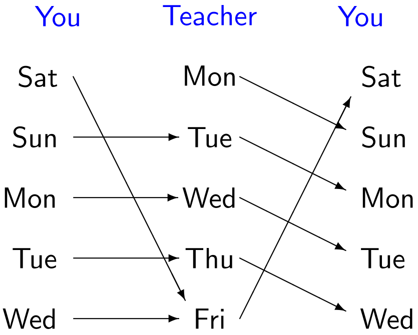
You

	Mon	Tue	Wed	Thu	Fri
Sat	$3, 2$	$2, 3$	$1, 4$	$0, 5$	$3, 6$
Sun	$-1, 6$	$3, 2$	$2, 3$	$1, 4$	$0, 5$
Mon	$0, 5$	$-1, 6$	$3, 2$	$2, 3$	$1, 4$
Tue	$0, 5$	$0, 5$	$-1, 6$	$3, 2$	$2, 3$
Wed	$0, 5$	$0, 5$	$0, 5$	$-1, 6$	$3, 2$

- On what days can you rationally start to study if you hold a simple belief hierarchy, and express common belief in rationality?



- We have seen:
- You can rationally choose Saturday or Wednesday under common belief in rationality with a simple belief hierarchy.
- Namely, the belief hierarchy that supports your choices Saturday and Wednesday is simple, as it is generated by the beliefs $\sigma_1 = \text{Sat}$ and $\sigma_2 = \text{Fri}$.



- Are there any other choices you can rationally make under common belief in rationality with a simple belief hierarchy?
- The beliefs diagram does not help here.
- Compute all Nash equilibria (σ_1, σ_2) in the game.

	Mon	Tue	Wed	Thu	Fri
Sat	3, 2	2, 3	1, 4	0, 5	3, 6
Sun	−1, 6	3, 2	2, 3	1, 4	0, 5
Mon	0, 5	−1, 6	3, 2	2, 3	1, 4
Tue	0, 5	0, 5	−1, 6	3, 2	2, 3
Wed	0, 5	0, 5	0, 5	−1, 6	3, 2

- Suppose that (σ_1, σ_2) is a Nash equilibrium.
- Step 1. Show that $\sigma_2(\mathit{Thu}) = 0$.
- Suppose that $\sigma_2(\mathit{Thu}) > 0$. Then, *Thu* must be optimal for the teacher under the belief σ_1 about your choice.
- This is only possible if $\sigma_1(\mathit{Wed}) > 0$.
- So, *Wed* must be optimal for you under the belief σ_2.
- This is only possible if $\sigma_2(\mathit{Fri}) = 1$. Contradiction.

	Mon	Tue	Wed	Thu	Fri
Sat	$3,2$	$2,3$	$1,4$	$0,5$	$3,6$
Sun	$-1,6$	$3,2$	$2,3$	$1,4$	$0,5$
Mon	$0,5$	$-1,6$	$3,2$	$2,3$	$1,4$
Tue	$0,5$	$0,5$	$-1,6$	$3,2$	$2,3$
Wed	$0,5$	$0,5$	$0,5$	$-1,6$	$3,2$

- Step 2. Show that $\sigma_2(Wed) = 0$.
- Suppose that $\sigma_2(Wed) > 0$. Then, *Wed* must be optimal for the teacher under the belief σ_1.
- This is only possible if $\sigma_1(Tue) > 0$.
- Then, *Tue* must be optimal for you under the belief σ_2.
- This is only possible if $\sigma_2(Thu) > 0$. Contradiction.

	Mon	Tue	Wed	Thu	Fri
Sat	$3, 2$	$2, 3$	$1, 4$	$0, 5$	$3, 6$
Sun	$-1, 6$	$3, 2$	$2, 3$	$1, 4$	$0, 5$
Mon	$0, 5$	$-1, 6$	$3, 2$	$2, 3$	$1, 4$
Tue	$0, 5$	$0, 5$	$-1, 6$	$3, 2$	$2, 3$
Wed	$0, 5$	$0, 5$	$0, 5$	$-1, 6$	$3, 2$

- Step 3. Show that $\sigma_2(Tue) = 0$.
- Suppose that $\sigma_2(Tue) > 0$. Then, *Tue* must be optimal for the teacher under the belief σ_1.
- This is only possible if $\sigma_1(Mon) > 0$. Otherwise, *Tue* would be strictly dominated for the teacher by $(0.9) \cdot Wed + (0.1) \cdot Thu$.
- So, *Mon* must be optimal for you under the belief σ_2.
- This is only possible if $\sigma_2(Wed) > 0$ or $\sigma_2(Thu) > 0$. Contradiction.

	Mon	Tue	Wed	Thu	Fri
Sat	$3,2$	$2,3$	$1,4$	$0,5$	$3,6$
Sun	$-1,6$	$3,2$	$2,3$	$1,4$	$0,5$
Mon	$0,5$	$-1,6$	$3,2$	$2,3$	$1,4$
Tue	$0,5$	$0,5$	$-1,6$	$3,2$	$2,3$
Wed	$0,5$	$0,5$	$0,5$	$-1,6$	$3,2$

- Step 4. Show that $\sigma_2(Mon) = 0$.
- Suppose that $\sigma_2(Mon) > 0$. Then, *Mon* must be optimal for the teacher under the belief σ_1.
- This is only possible if $\sigma_1(Sun) > 0$. Otherwise, *Mon* would be strictly dominated for the teacher by $(0.9) \cdot Tue + (0.09) \cdot Wed + (0.01) \cdot Thu$.
- So, *Sun* must be optimal for you under the belief σ_2.
- This is only possible if $\sigma_2(Tue) > 0$. Contradiction.

	Mon	Tue	Wed	Thu	Fri
Sat	$3,2$	$2,3$	$1,4$	$0,5$	$3,6$
Sun	$-1,6$	$3,2$	$2,3$	$1,4$	$0,5$
Mon	$0,5$	$-1,6$	$3,2$	$2,3$	$1,4$
Tue	$0,5$	$0,5$	$-1,6$	$3,2$	$2,3$
Wed	$0,5$	$0,5$	$0,5$	$-1,6$	$3,2$

- So, if (σ_1, σ_2) is a Nash equilibrium, then σ_2 must assign probability 0 to Mon, Tue, Wed and Thu. Hence, $\sigma_2 = Fri$.
- But then, your optimal choices under the belief σ_2 are Sat and Wed.
- Hence, your only Nash choices in this game are Sat and Wed.
- These are the only choices you can rationally make under common belief in rationality with a simple belief hierarchy.

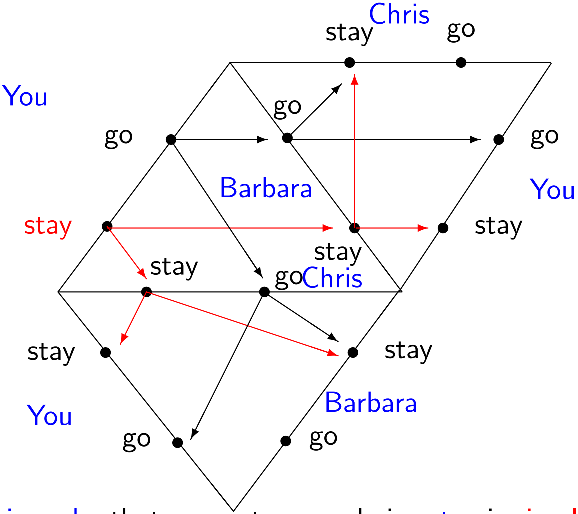
	Mon	Tue	Wed	Thu	Fri
Sat	$3,2$	$2,3$	$1,4$	$0,5$	$3,6$
Sun	$-1,6$	$3,2$	$2,3$	$1,4$	$0,5$
Mon	$0,5$	$-1,6$	$3,2$	$2,3$	$1,4$
Tue	$0,5$	$0,5$	$-1,6$	$3,2$	$2,3$
Wed	$0,5$	$0,5$	$0,5$	$-1,6$	$3,2$

Summarizing

- Under common belief in rationality, you can rationally start to study on any day between Saturday and Wednesday.
- However, if you hold a simple belief hierarchy in addition, then under common belief in rationality you can only rationally start to study on Saturday or Wednesday.
- Crucial difference: With a simple belief hierarchy, you believe that the teacher is correct about your beliefs.

Example: Movie or party?

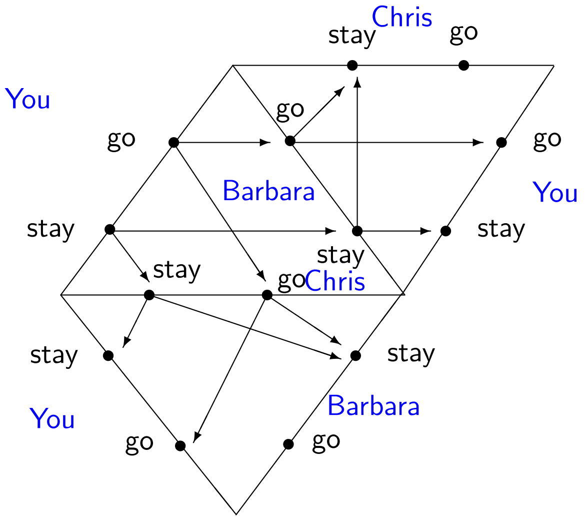
- Having a good time at the party gives you utility 3, watching the movie gives you utility 2, whereas having a bad time at the party gives you utility 0. Similarly for Barbara and Chris.
- You will only have a good time at the party if Barbara and Chris both join.
- Barbara will only have a good time at the party if you join, but not Chris.
- Chris will only have a good time at the party if you join, but not Barbara.
- What choice(s) can you rationally make if you hold a simple belief hierarchy, and express common belief in rationality?



- The belief hierarchy that supports your choice stay is simple: It is completely generated by the beliefs

$$\sigma_1 = \text{You stay},\ \sigma_2 = \text{Barbara stays},\ \sigma_3 = \text{Chris stays.}$$

- So, you can rationally stay at home under common belief in rationality with a simple belief hierarchy.



- In this beliefs diagram, your choice to go the party is not supported by a simple belief hierarchy.
- But can your choice go be supported by a simple belief hierarchy that expresses common belief in rationality?

- Let us try to find all Nash equilibria in this game, and see whether your choice go is a Nash choice.

You stay	C stays	C goes
B stays	2, 2, 2	2, 2, 0
B goes	2, 0, 2	2, 0, 0

You go	C stays	C goes
B stays	0, 2, 2	0, 2, 3
B goes	0, 3, 2	3, 0, 0

- Suppose that $(\sigma_1, \sigma_2, \sigma_3)$ is a Nash equilibrium in this game.
- We first show that $\sigma_1(go) = 0$.
- Assume that $\sigma_1(go) > 0$. Then, *go* must be optimal for you under the belief (σ_2, σ_3).
- For you, $u_1(go) = 3 \cdot \sigma_2(go) \cdot \sigma_3(go)$, whereas $u_1(stay) = 2$.
- Hence, $\sigma_2(go) \cdot \sigma_3(go) \geq 2/3$, which implies $\sigma_2(go) \geq 2/3$ and $\sigma_3(go) \geq 2/3$. This implies $\sigma_3(stay) \leq 1/3$.
- So, *go* must be optimal for Barbara under the belief (σ_1, σ_3).
- But for Barbara,

$$u_2(go) = 3 \cdot \sigma_1(go) \cdot \sigma_3(stay) \leq 1 < u_2(stay),$$

which means that *go* is not optimal for Barbara. Contradiction.

You stay	C stays	C goes
B stays	2, 2, 2	2, 2, 0
B goes	2, 0, 2	2, 0, 0

You go	C stays	C goes
B stays	0, 2, 2	0, 2, 3
B goes	0, 3, 2	3, 0, 0

- So we conclude that $\sigma_1(stay) = 1$.
- But then, for Barbara only stay can be optimal under the belief (σ_1, σ_3). Hence, $\sigma_2 = stay$.
- Similarly, for Chris only stay can be optimal under the belief (σ_1, σ_2). Consequently, $\sigma_3 = stay$.
- So, the only Nash equilibrium is

$$\sigma_1 = stay, \ \sigma_2 = stay, \ \sigma_3 = stay.$$

- Under the belief (σ_2, σ_3), your only optimal choice is to stay at home. Hence, your only Nash choice is to stay at home.

You stay	C stays	C goes
B stays	2, 2, 2	2, 2, 0
B goes	2, 0, 2	2, 0, 0

You go	C stays	C goes
B stays	0, 2, 2	0, 2, 3
B goes	0, 3, 2	3, 0, 0

Summarizing

- Under common belief in rationality you can either stay at home, or go to the party.
- However, if you hold a simple belief hierarchy, then under common belief in rationality your only rational choice is to stay at home.
- Crucial difference: With a simple belief hierarchy, you believe that Barbara has the same belief about Chris' choice as you do.

Conditions leading to simple belief hierarchies

- We have concentrated on simple belief hierarchies.
- But which epistemic conditions lead to a simple belief hierarchy?
- We focus on the case of two players only.

- In a two-player game, a simple belief hierarchy for player i is completely generated by a pair of beliefs (σ_i, σ_j). That is:

- player i holds belief σ_j about j's choice,
- player i believes that player j holds belief σ_i about i's choice,
- player i believes that player j believes that, indeed, player i holds belief σ_j about j's choice,
- player i believes that player j believes that player i believes that, indeed, player j holds belief σ_i about i's choice,
- and so on.

- So, if player i holds a simple belief hierarchy, then he believes that his opponent is correct about his belief hierarchy. We say that player i believes that player j holds correct beliefs.

- Moreover, if player i holds a simple belief hierarchy, he also believes that player j believes that i has correct beliefs.

Definition (Belief that opponents hold correct beliefs)

A type t_i believes that his opponent holds correct beliefs if he believes that his opponent believes that, indeed, his type is t_i.

- We have seen that in a two-player game, a type with a simple belief hierarchy believes that his opponent holds correct beliefs, and believes that his opponent believes that he himself holds correct beliefs too.
- In fact, the other direction is also true: If in a two-player game a type believes that his opponent holds correct beliefs, and believes that his opponent believes that he himself holds correct beliefs too, then this type has a simple belief hierarchy.

Theorem (Characterization of types with a simple belief hierarchy in two-player games)

Consider a game with two players.

A type t_i for player i has a simple belief hierarchy, if and only if, t_i believes that his opponent holds correct beliefs, and believes that his opponent believes that he himself holds correct beliefs too.

- Proof. Suppose that type t_i believes that his opponent holds correct beliefs, and believes that his opponent believes that he himself holds correct beliefs too.
- Show: Type t_i assigns probability 1 to a single type t_j for player j.
- Suppose that t_i would assign positive probability to two different types t_j and t_j' for player j.

$$
\begin{tikzcd}
& t_j \arrow[dr] & & t_j \\
t_i \arrow[ur] \arrow[dr] & & t_i \arrow[ur] \arrow[dr] & \\
& t_j' \arrow[ur] & & t_j'
\end{tikzcd}
$$

- Then, t_j would not believe that i holds correct beliefs. Contradiction.

Theorem (Characterization of types with a simple belief hierarchy in two-player games)

Consider a game with two players.

A type t_i for player i has a simple belief hierarchy, if and only if, t_i believes that his opponent holds correct beliefs, and believes that his opponent believes that he himself holds correct beliefs too.

- So, we know that t_i assigns probability 1 to some type t_j for player j, and t_j assigns probability 1 to t_i.
- Let σ_j be the belief that t_i has about j's choice, and let σ_i be the belief that t_j has about i's choice.

$$t_i \xrightarrow[\sigma_j]{} t_j \xrightarrow[\sigma_i]{} t_i$$

- But then, t_i's belief hierarchy is generated by (σ_i, σ_j). So, t_i has a simple belief hierarchy. ∎

- Be careful: If we have more than two players, then these conditions are no longer enough to induce simple belief hierarchies.
- In a game with more than two players, we need to impose the following extra conditions:
- you believe that player j has the same belief about player k as you do;
- your belief about player j's choice must be independent from your belief about player k's choice.

How reasonable is Nash equilibrium?

- We have seen that a Nash equilibrium makes the following assumptions:
- you believe that your opponents are correct about the beliefs that you hold;
- you believe that player *j* holds the same belief about player *k* as you do;
- your belief about player *j*'s choice is independent from your belief about player *k*'s choice.
- Each of these conditions is actually very questionable.
- Therefore, Nash equilibrium is not such a natural concept after all.
- Common belief in rationality is a much more natural concept.