Why Forward Induction leads to the Backward Induction Outcome: A New Proof for Battigalli's Theorem

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Abstract

Battigalli (1997) has shown that the forward induction concept of *extensive-form rationalizability* leads to the backward induction outcome in every dynamic game with perfect information and without relevant ties. In this paper we present a new proof for this remarkable result, based on the notion of *choice sequences* (von Stengel (1996)). For a given player, a choice sequence is the sequence of choices for that player on the path from the root to some history in the game. Based on this notion we present a recursive procedure, called *iterated elimination of all strictly dominated choice sequences,* which characterizes for every round k those outcomes that are possible at round k of the extensive-form rationalizability procedure. We show, moreover, that this procedure is order independent, and that the backward induction procedure can be mimicked by a specific order of elimination. These results imply Battigalli's theorem.

Keywords: Backward induction, forward induction, extensive-form rationalizability, Battigalli's theorem, choice sequences, order independence.

JEL Classification: C72, C73.

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1 Introduction

Backward induction and forward induction are two fundamentally different lines of reasoning in dynamic games. In backward induction, a player believes throughout the game that his opponents will choose rationally in the future, regardless of what these opponents have done in the past. This principle is the basis for the well-known backward induction procedure in dynamic games with perfect information, and the concept of common belief in future rationality (Perea (2014), see also Penta (2016) and Baltag, Smets and Zvesper (2009) for related lines of reasoning) for general dynamic games. The backward induction principle is also implicitly present in equilibrium concepts like subgame perfect equilibrium (Selten (1965)) and sequential equilibrium (Kreps and Wilson (1982)). A common feature of all these backward induction concepts is thus that players are not required to reason about the opponents' past choices, but instead are required to believe that the opponents will act rationally in the future independent of what these opponents have done in the past.

Forward induction, on the other hand, *does* require the players to actively reason about the opponents' past choices. Although there is no unique definition of forward induction in the literature, the main idea is that a player, whenever possible, tries to interpret the opponent's past moves as being part of a rational strategy, and that he bases his belief about the opponent's *future* moves on this hypothesis. *Extensive-form rationalizability* (Pearce (1984), Battigalli (1997)) is a very basic and natural forward induction concept, based on the idea that a player, whenever possible, must believe that his opponents are implementing rational strategies. This idea can be formalized by the epistemic condition of *strong belief in the opponents' rationality* (Battigalli and Siniscalchi (2002)), which provides the basis for *common strong belief in rationality* – a concept that characterizes extensive-form rationalizability on an epistemic level.

More precisely, extensive-form rationalizability is an inductive procedure that iteratedly removes strategies and conditional belief vectors from the game. In the first round we remove all strategies that can never be rational for any conditional belief vector, and subsequently restrict to conditional belief vectors that – whenever possible – assign probability one to the strategies that have survived this first elimination. This restriction thus mimicks the condition of strong belief in the opponents' rationality. In the second round we then eliminate all strategies that can never be rational for any conditional belief vector that has survived the first round, and subsequently restrict the conditional belief vectors even further by focusing on those that – whenever possible – assign probability one to the strategies that have survived the second elimination, and so on.

Extensive-form rationalizability, being a forward induction concept, is based on a completely different line of reasoning than backward induction. Indeed, extensive-form rationalizability requires players to critically reason about the opponents' past choices in the game, whereas backward induction does not. Despite this fundamental difference, Battigalli (1997) shows in his Theorem 4 that both lines of reasoning uniquely lead to the backward induction outcome in dynamic games with perfect information and without relevant ties. This result is remarkable, as there is no obvious reason to expect that these two lines of reasoning – which are so fundamentally different – would lead to exactly the same outcome in this class of games. The difference is also illustrated by the fact that both lines of reasoning may lead to different choices for the players *off the backward induction path* – something we will illustrate in Section 3. At the same time, Battigalli's theorem is crucial for the foundations of game theory, as backward induction and forward induction both play a prominent role in the theory of dynamic games.

It therefore seems important for game theorists to not only know *that* Battigalli's theorem holds, but also to know *why* it holds. The purpose of this paper is to make a step forward in that direction, by delivering a new proof for Battigalli's theorem which – we hope – leads to an even better understanding of *why* it holds.

Our proof consists of three parts. In the first part we present a recursive elimination procedure which, at every round k, characterizes exactly those *outcomes* in the game that are possible given the strategies that survive round k of the extensive-form rationalizability procedure. In particular, the output of this procedure yields exactly the extensive-form rationalizable outcomes in the game. In the second part we show that the output of our procedure is *order independent* – that is, the final output does not depend on the order or speed by which we perform the eliminations. In the third part we show that the backward induction procedure can be mimicked by our procedure if we choose a specific order of elimination. By the order independence property, this alternative order of elimination – which leads to the backward induction outcome – must lead to the same outcome as the original procedure. Since the original procedure characterizes the extensive-form rationalizable outcomes, it follows that the unique extensive-form rationalizable outcome is the backward induction outcome. Hence, Battigalli's theorem follows.

The elimination procedure that we use in the first part is based on the elimination of *choice sequences* (von Stengel (1996)) – rather than strategies – from the game. A choice sequence for a player is the sequence of choices for that player on the path from the root to some (non-terminal or terminal) history in the game. It can thus be viewed as a partial description of a strategy, by focusing only on those choices that lie on the same path from the root to some history in the game. We say that a choice sequence σ_i for player i is *strictly dominated* at some history h where player i is active, if the highest utility that i can obtain from h onwards by choosing in accordance with σ_i is lower than the maximum utility that i can guarantee for himself if the game were to start at h. In Theorem 4.1 we show that a choice sequence is part of a rational strategy – that is, can be extended to a strategy that is optimal for some conditional belief vector at every relevant history – if and only if the choice sequence is not strictly dominated. This result turns out to be the key to our proof.

The elimination procedure that we develop in the first part works as follows. In the first round we start by eliminating all strictly dominated choice sequences from the game, and subsequently restrict the game to those histories that can still be reached under the choice sequences that have survived the first round. We thus obtain a smaller, reduced game. Within this reduced game, we again eliminate all strictly dominated choice sequences in round 2, and subsequently reduce the game even further by restricting to those histories that can still be reached under the choice sequences that have survived the second round. We proceed in this way until no remaining choice sequences are strictly dominated. We call this procedure the *iterated elimination of all strictly dominated choice sequences.* In Section 5 we show, with the help of Theorem 4.1, that the reduced game obtained in round k of the procedure contains exactly those outcomes that are possible under the strategies that survive round k of the extensive-form rationalizability procedure. As a consequence, the reduced game obtained at the end of the procedure contains precisely the extensive-form rationalizable outcomes in the game.

In Section 6 we show that the iterated elimination of all strictly dominated choice sequences is order independent. More precisely, we show the following: If we remove a strictly dominated choice sequence σ_i from the game such that, at the last history where σ_i is defined there is at least on other choice available, then it does not matter for the final output whether we apply the procedure to the original game or to the reduced game that results from removing σ_i. Strictly dominated choice sequences of this kind are called *regular.* By an iterated application of this property, it thus follows that the output of the procedure remains unaffected if we successively eliminate regular and strictly dominated choice sequences in an arbitrary fashion.

In Section 7 we finally show that the backward induction procedure can be mimicked by the successive elimination of regular and strictly dominated choice sequences in a specific order. By the order independence property above it then follows that the original procedure must yield the same output as this alternative procedure, which obviously leads to the backward induction outcome. Since the original procedure characterizes the extensive-form rationalizable outcomes, we conclude that the unique extensive-form rationalizable outcome must be the backward induction outcome. This proves Battigalli's theorem.

This paper is not the first to prove Battigalli's theorem. Much credit should of course go to Battigalli (1997), who was the first to prove this result by relying on certain properties of fully stable sets (Kohlberg and Mertens (1986)). The result also follows from Chen and Micali (2013), who show that the *outcomes* that are selected by the iterated conditional dominance procedure (Shimoji and Watson (1998)) do not depend on the specific order or speed of elimination. Since Shimoji and Watson (1998) show that the iterated conditional dominance procedure characterizes the extensive-form rationalizable strategies, and the backward induction outcome can be obtained by a specific order of elimination in this procedure, it follows from Chen and Micali (2013) that extensive-form rationalizability uniquely leads to the backward induction outcome. Heifetz and Perea (2015), finally, prove Battigalli's theorem via a different route. The main step in their proof is to show that the extensive-form rationalizable outcomes of a game do not change if we *truncate* the game, by eliminating the suboptimal choices at every last non-terminal history. The main difference between our proof and the proofs above is our use of *choice sequences,* rather than strategies, and the fact that we use these choice sequences to derive a characterization of the outcomes that are possible at every round of the extensive-form rationalizability procedure.

The outline of this paper is as follows. In Section 2 we present the model of dynamic games with perfect information. In Section 3 we define the backward induction procedure and the extensive-form rationalizability procedure, and show that the *strategies* selected by the extensive-form rationalizability procedure depend on the order and speed by which we eliminate strategies from the game. In Section 4 we introduce the notion of a choice sequence, which plays such a crucial role in this paper, and show that a choice sequence is part of rational strategy if and only if the choice sequence is not strictly dominated. In Section 5 we present our procedure called *iterated elimination of all strictly dominated choice sequences,* and prove that this procedure characterizes, for every round k, the outcomes that are possible under round k of the extensive-form rationalizability procedure. In Section 6 we show that this procedure is order independent in the sense described above. In Section 7 we use all insights gathered in Sections 5 and 6 to prove Battigalli's theorem. We provide some concluding remarks in Section 8. The longer proofs are all gathered in the appendix, whereas the shorter proofs are given in the main body of this paper. However, for each of the results requiring a longer proof we give a sketch of the proof in the main body. By doing so, we hope that by reading the main body of this paper the reader will get a clear intuition for why Battigalli's theorem holds.

2 Dynamic Games with Perfect Information

As we assume most readers will be familiar with the model of a dynamic game with perfect information, we only introduce the necessary notation here.

A *finite dynamic game with perfect information* is a tuple

$$G = (I, H, Z, (H_i)_{i \in I}, (C_i(h))_{i \in I, h \in H_i}, (u_i)_{i \in I})$$

where

(a) $I = \{1, 2, ..., n\}$ is the finite set of players;

(b) H is the finite set of non-terminal histories, representing the situations where one of the players must make a choice. By $\emptyset$ we denote the *root* of the game, which is the non-terminal history where the game starts;

(c) Z is the finite set of terminal histories, representing the possible outcomes of the game;

(d) H_i is the set of non-terminal histories where player i must make a choice. In a perfect information game, exactly one player moves at every non-terminal history. Hence we require that $H = \cup_{i \in I} H_i$ and $H_i \cap H_j = \emptyset$ whenever $i \neq j$;

(e) $C_i(h)$ is the finite set of choices available to player i at history $h \in H_i$; and

(f) $u_i : Z \to \mathbb{R}$ is player i's utility function, assigning to every terminal history $z \in Z$ some utility $u_i(z)$.

For every non-terminal history $h \in H_i$ and choice $c_i \in C_i(h)$, we denote by hc_i the (terminal or non-terminal) history that immediately follows choice c_i at h. We say that a history $h \in H \cup Z$ *follows* another history $h' \in H$ if there is a sequence of choices $c^1, c^2, ..., c^K$ such that $h = h'c^1c^2...c^K$. A history $h \in H \cup Z$ is said to *weakly follow* $h' \in H \cup Z$ if either h follows h'

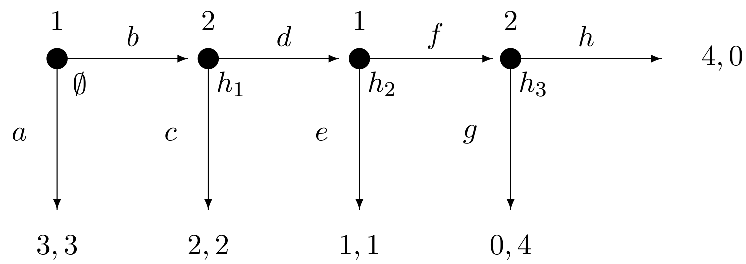

Figure 1: Reny's game

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or $h = h'$. In the obvious way, we can then also define what it means for h to (*weakly*) *precede* another history h'. Often we identify a history $h \in H \cup Z$ with the sequence of choices $c^1 c^2 ... c^K$ on the path from $\emptyset$ to h.

The game G is said to be *without relevant ties* (Battigalli (1997)) if for every player i, every $h \in H_i$, every two distinct choices $c_i, c_i' \in C_i(h)$, every $z \in Z$ weakly following hc_i, and every $z' \in Z$ weakly following hc_i', it holds that $u_i(z) \neq u_i(z')$. Hence, two different choices for player i always lead to different utilities for player i at the end.

We view a strategy for player i as a *plan of action* (Rubinstein (1991)), assigning choices only to those histories $h \in H_i$ that are not precluded by previous choices. Formally, consider a set of non-terminal histories $\hat{H}_i \subseteq H_i$, and a mapping $s_i : \hat{H}_i \to \cup_{h \in \hat{H}_i} C_i(h)$ assigning to every history $h \in \hat{H}_i$ some available choice $s_i(h) \in C_i(h)$. We say that a history $h \in H$ is *reachable* under s_i if at every history $h' \in \hat{H}_i$ preceding h, the choice $s_i(h')$ is the unique choice that leads to h. The mapping $s_i : \hat{H}_i \to \cup_{h \in \hat{H}_i} C_i(h)$ is called a *strategy* if $\hat{H}_i$ contains exactly those histories in H_i that are reachable under s_i.

By S_i we denote the set of strategies for player i. For every history $h \in H$ and player i, we denote by $S_i(h)$ the set of strategies for player i under which h is reachable.

As an illustration, consider the game G in Figure 1, which is based on Figure 3 in Reny (1992). It is easily seen that G is a finite dynamic game with perfect information and without relevant ties. The non-terminal histories are $\emptyset, h_1, h_2$ and h_3, the strategies for player 1 are $a, (b, e)$ and (b, f), whereas the strategies for player 2 are $c, (d, g)$ and (d, h). We also have, for instance, that $S_1(h_1) = \{(b, e), (b, f)\}$ as h_1 is only reachable if player 1 chooses b at $\emptyset$.

3 Backward Induction and Extensive-Form Rationalizability

In this section we will introduce the well-known backward induction procedure and the extensive-form rationalizability procedure (Pearce (1984), Battigalli (1997)). The extensive-form rationalizability procedure recursively eliminates, at every round, some strategies and conditional belief vectors for the players. We then provide a sufficient condition which guarantees that a strategy will survive round k of the extensive-form rationalizability procedure. This condition will be important for proving Battigalli's theorem. At the end of this section we will illustrate, by means of the example in Figure 1, that the final output of the extensive-form rationalizability procedure depends on the order and speed by which we eliminate strategies from the game.

3.1 Backward Induction

Consider a finite dynamic game G with perfect information and without relevant ties. For every history $h \in H \cup Z$, let $d(h)$ be the maximal number of consecutive choices between h and a terminal history. We call $d(h)$ the *degree* of history h. We define for every $h \in H \cup Z$ the *backward induction utilities* $u_i^{bi}(h)$ and for every $h \in H_i$ the *backward induction choice* $c_i^{bi}(h)$, by induction on the degree $d(h)$.

Consider first a history h of degree 0, meaning that h is a terminal history. We define, for all players i,

$$u_i^{bi}(h) := u_i(h).$$

Now, let $k \geq 1$, and assume that $u_i^{bi}(h)$ has been defined for all players i and all histories $h \in H \cup Z$ with degree $d(h) \leq k-1$. Take an arbitrary history $h \in H_i$ with degree k. Let $c_i^{bi}(h)$ be a choice in $C_i(h)$ such that

$$u_i^{bi}(hc_i^{bi}(h)) = \max_{c_i \in C_i(h)} u_i^{bi}(hc_i).$$

For every player $j \in I$, define

$$u_j^{bi}(h) := u_j^{bi}(hc_i^{bi}(h)).$$

Since the game G is without relevant ties, the backward induction choice $c_i^{bi}(h)$ is unique for every player i and every $h \in H_i$. By induction on k, this procedure then defines the unique backward induction choice $c_i^{bi}(h)$ for every player i and every history $h \in H_i$.

The terminal history z^{bi} that is reached by the combination of backward induction choices $(c_i^{bi}(h))_{i \in I, h \in H_i}$ is called the *backward induction outcome* in G.

In the game of Figure 1, it is easily seen that the backward induction choices are a, c, e and g, and that therefore the unique backward induction outcome is the terminal history a.

3.2 Extensive-Form Rationalizability

The *extensive-form rationalizability* procedure has been defined by Pearce (1984), and later simplified by Battigalli (1997). It recursively eliminates strategies and conditional belief vectors from the game, where in every step an elimination of strategies is followed by an elimination of conditional belief vectors. In order to formally state this procedure for games with perfect information, we need some additional definitions.

For a finite set X, we denote by $\Delta(X)$ the set of probability distributions on X. For a player i and history $h \in H_i$, let $S_{-i}(h) := \times_{j \neq i} S_j(h)$ be the set of opponents' strategy combinations under which h is reachable.

A *conditional belief vector* for player i is tuple $b_i = (b_i(h))_{h \in H_i}$ where $b_i \in \Delta(S_{-i}(h))$ for every $h \in H_i$. Here, $b_i(h)$ represents the conditional probabilistic belief that i holds at h about the opponents' strategy choices. We say that the conditional belief vector b_i satisfies *Bayesian updating* is for every $h, h' \in H_i$ where h' follows h and $b_i(h)(S_{-i}(h')) > 0$, it holds that

$$b_i(h')(s_{-i}) = \frac{b_i(h)(s_{-i})}{b_i(h)(S_{-i}(h'))} \text{ for all } s_{-i} \in S_{-i}(h').$$

By B_i we denote the set of conditional belief vectors for player i that satisfy Bayesian updating.

For a given conditional belief vector b_i and a set $E \subseteq S_{-i}$ of opponents' strategy combinations, we say that b_i *strongly believes* E if $b_i(h)(E) = 1$ for all $h \in H_i$ where $S_{-i}(h) \cap E \neq \emptyset$. That is, b_i assigns full probability to E at all histories $h \in H_i$ where E is logically consistent with the event that h has been reached.

For a strategy combination $s = (s_i)_{i \in I}$ we denote by $z(s)$ the induced terminal history. For a history $h \in H_i$, a strategy $s_i \in S_i(h)$, and a conditional belief $b_i(h) \in \Delta(S_{-i}(h))$, we denote by

$$u_i(s_i, b_i(h)) := \sum_{s_{-i} \in S_{-i}(h)} b_i(h)(s_{-i}) \cdot u_i(z(s_i, s_{-i}))$$

the induced expected utility at h. We say that strategy s_i is *rational* for belief $b_i(h)$ at h if $u_i(s_i, b_i(h)) \geq u_i(s_i', b_i(h))$ for all $s_i' \in S_i(h)$. That is, strategy s_i yields the highest possible expected utility at h under the belief $b_i(h)$. We say that strategy s_i is rational for a conditional belief vector $b_i = (b_i(h))_{h \in H_i}$ if s_i is rational for the belief $b_i(h)$ at every $h \in H_i$ that is reachable under s_i.

Definition 3.1 (Extensive-Form Rationalizability) *Consider a finite dynamic game G with perfect information.*

(Induction start) *Set $S_i^0 := S_i$ and $B_i^0 := B_i$ for all players i.*

(Induction step) *Let $k \geq 1$, and assume that S_i^{k-1} and B_i^{k-1} have already been defined for all players i. Then, define for all players i*

$$\begin{aligned} S_i^k &:= \{s_i \in S_i \mid s_i \text{ rational for some } b_i \in B_i^{k-1}\}, \\ B_i^k &:= \{b_i \in B_i^{k-1} \mid b_i \text{ strongly believes } S_{-i}^k\}. \end{aligned}$$

A strategy $s_i \in S_i$ is called ***extensive-form rationalizable*** *if $s_i \in S_i^k$ for all $k \geq 0$.*

Here, by S_{-i}^k we denote the set $\times_{j \neq i} S_j^k$. By $S_i^\infty := \cap_{k \geq 0} S_i^k$ we denote the set of extensive-form rationalizable strategies for player i. We call an outcome $z \in Z$ *extensive-form rationalizable* if there is a strategy combination $(s_i)_{i \in I}$ in $\times_{i \in I} S_i^\infty$ that induces z.

As an illustration, consider again the game G from Figure 1. It may be verified that

$$S_1^1 = \{a, (b, f)\} \text{ and } S_2^1 = \{c, (d, g)\}.$$

Note that strategy (b, e) can never be rational for player 1 for any conditional belief vector, since (b, e) yields player 1 at most utility 2 at $\emptyset$ whereas player 1 can guarantee utility 3 there by choosing a. Similarly, strategy (d, h) is never rational for player 2 for any conditional belief vector, as the choice h is suboptimal for player 2 at h_3. By construction, we then have that

$$\begin{aligned} B_1^1 &= \{b_1 \in B_1^0 \mid b_1 \text{ strongly believes } \{c, (d, g)\}\} \\ &= \{b_1 \in B_1^0 \mid b_1(\emptyset)(\{c, (d, g)\}) = 1 \text{ and } b_1(h_2)(\{(d, g)\}) = 1\} \end{aligned}$$

and

$$\begin{aligned} B_2^1 &= \{b_2 \in B_2^0 \mid b_2 \text{ strongly believes } \{a, (b, f)\}\} \\ &= \{b_2 \in B_2^0 \mid b_2(h_1)(\{(b, f)\}) = b_2(h_3)(\{(b, f)\}) = 1\}. \end{aligned}$$

Note that a is the only strategy for player 1 that is rational for a conditional belief vector in B_1^1. Similarly, (d, g) is the only strategy for player 2 that is rational for the unique conditional belief vector in B_2^1. Hence,

$$S_1^2 = \{a\} \text{ and } S_2^2 = \{(d, g)\},$$

which implies that

$$\begin{aligned} B_1^2 &= \{b_1 \in B_1^1 \mid b_1 \text{ strongly believes } \{(d, g)\}\} \\ &= \{b_1 \in B_1^0 \mid b_1(\emptyset)(\{(d, g)\}) = b_1(h_2)(\{(d, g)\}) = 1\} \end{aligned}$$

and

$$B_2^2 = \{b_2 \in B_2^1 \mid b_2 \text{ strongly believes } \{a\}\} = B_2^1.$$

After this round the procedure terminates, as $S_1^3 = S_1^2$ and $S_2^3 = S_2^2$. Hence, the extensive-form rationalizable strategies are a for player 1 and (d, g) for player 2, which implies that the unique extensive-form rationalizable outcome is the terminal history a. We thus conclude that the unique extensive-form rationalizable outcome is the same as the backward induction outcome in this game G.

The "forward induction story" behind the eliminations above is as follows: If player 2 observes at h_1 that player 1 has chosen b, he tries to interpret b as being part of a rational strategy for player 1. Therefore, player 2 must believe at h_1 that player 1 will choose f at h_2, as that is the only way for player 1 to obtain more than 3 – the utility he could have guaranteed by choosing a at $\emptyset$. This argument is mimicked by the set of beliefs B_2^1 above. If player 2 reasons in this way, his best strategy is to choose (d, g), which is player 2's only strategy in S_2^2. Player 1, anticipating on player 2 choosing (d, g), will therefore choose a.

Hence, the reason that player 1 chooses a in extensive-form rationalizability is that he expects player 2 to choose d and g if he were to choose b instead of a at $\emptyset$. In contrast, the reason that player 1 chooses a in the backward induction procedure is that he expects player 2 to choose c if he were to choose b instead of a at $\emptyset$. We thus see that these two fundamentally different lines of reasoning lead to the same outcome a in this game, but for different reasons.

3.3 Sufficient Condition for Optimality

Consider the extensive-form rationalizability procedure above. By definition, the set S_i^k at round k consists of those strategies s_i that are rational for some conditional belief vector b_i in B_i^{k-1}. That is, for every $h \in H_i$ that is reachable under s_i, it must hold that

$$u_i(s_i, b_i(h)) \geq u_i(s_i', b_i(h)) \text{ for all } s_i' \in S_i(h).$$

In the following theorem we show that it is sufficient to check the above inequality for all $s_i' \in S_i^{k-1} \cap S_i(h)$.

Theorem 3.1 (Sufficient condition for optimality) *Let the sets S_i^k and B_i^k be defined as in the extensive-form rationalizability procedure. Then, $s_i \in S_i^k$, if and only if, there is some $b_i \in B_i^{k-1}$ such that*

$$u_i(s_i, b_i(h)) \geq u_i(s_i', b_i(h)) \textit{ for all } s_i' \in S_i^{k-1} \cap S_i(h)$$

at all $h \in H_i$ that are reachable under s_i.

The proof can be found in the appendix. The key idea is the following. Suppose, contrary to what we want to show, that $u_i(s_i, b_i(h)) \geq u_i(s_i', b_i(h))$ for all strategies $s_i' \in S_i^{k-1} \cap S_i(h)$, but that $u_i(s_i, b_i(h)) < u_i(s_i', b_i(h))$ for some strategy $s_i' \in S_i(h)$. Then, we can construct another strategy $s_i'' \in S_i(h)$ that is rational for the conditional belief vector b_i. Hence, by definition, s_i'' will be in S_i^k – and therefore in S_i^{k-1} – and $u_i(s_i', b_i(h)) \leq u_i(s_i'', b_i(h))$. It thus follows that $u_i(s_i, b_i(h)) < u_i(s_i'', b_i(h))$ where $s_i'' \in S_i^{k-1} \cap S_i(h)$, which is a contradiction to the assumption above.

Theorem 3.1 will play a crucial role in the proof of Theorem 5.1, where we characterize for every k those outcomes that are reachable under the strategies that survive round k of the extensive-form rationalizability procedure.

3.4 Order Dependence

Many elimination procedures in game theory have the property that the order or speed of elimination do not affect the final output. Examples are the iterated elimination of strictly dominated choices in static games, or the backward dominance procedure for dynamic games, which characterizes those strategies that can rationally be chosen if the players express common belief in future rationality (see Perea (2014)).

Things are different for the extensive-form rationalizability procedure: We will show, by means of the game in Figure 1, that the output of this procedure can change if we alter the order or speed of elimination.

Consider again the game G from Figure 1. Suppose that in round 1 of the procedure we would only eliminate strategy (d,h) for player 2, but not strategy (b,e) for player 1. That is,

$$\hat{S}_1^1 = S_1 = \{a,(b,e),(b,f)\} \text{ and } \hat{S}_2^1 = \{c,(d,g)\}.$$

Then, the induced sets of conditional belief vectors would be

$$\begin{aligned}\hat{B}_1^1 &= \{b_1 \in B_1^0 \mid b_1 \text{ strongly believes } \{c,(d,g)\}\} \\ &= \{b_1 \in B_1^0 \mid b_1(\emptyset)(\{c,(d,g)\}) = 1 \text{ and } b_1(h_2)(\{(d,g)\}) = 1\}\end{aligned}$$

and

$$\hat{B}_2^1 = \{b_2 \in B_2^0 \mid b_2 \text{ strongly believes } S_1\} = B_2^0.$$

Note that strategies (b,e) and (b,f) are never rational for player 1 for any conditional belief vector in $\hat{B}_1^1$. Suppose that in round 2, we only eliminate strategy (b,f) but not (b,e) for player 1, and that we do not eliminate any further strategy for player 2. This leads to strategy sets

$$\hat{S}_1^2 = \{a,(b,e)\} \text{ and } \hat{S}_2^2 = \{c,(d,g)\}$$

which induces the sets of conditional belief vectors

$$\hat{B}_1^2 = \{b_1 \in \hat{B}_1^1 \mid b_1 \text{ strongly believes } \{c,(d,g)\}\} = \hat{B}_1^1$$

and

$$\begin{aligned}\hat{B}_2^2 &= \{b_2 \in \hat{B}_2^1 \mid b_2 \text{ strongly believes } \{a,(b,e)\}\} \\ &= \{b_2 \in B_2^0 \mid b_2(h_1)(\{(b,e)\}) = 1\}.\end{aligned}$$

Note that (d,g) is not rational for player 2 at h_1 for any conditional belief vector b_2 in $\hat{B}_2^2$, due to the fact that $b_2(h_1)(\{(b,e)\}) = 1$. As strategy (b,e) is not rational for player 1 for any conditional belief vector in $\hat{B}_1^2$, we can eliminate strategies (b,e) and (d,g) in round 3, and obtain

$$\hat{S}_1^3 = \{a\} \text{ and } \hat{S}_2^3 = \{c\}.$$

Hence, the particular order of elimination we have chosen uniquely yields the strategy c for player 2, which is different from the strategy (d, g) we obtained under the original extensive-form rationalizability procedure. We thus conclude that changing the order and speed of elimination may change the output of the extensive-form rationalizability procedure.

Recall that the objective of this paper is to prove Battigalli's theorem, which states that in every finite dynamic game with perfect information and without relevant ties, the unique extensive-form rationalizable outcome is the backward induction outcome. One difficulty in proving this result is precisely the fact that the output of the extensive-form rationalizability procedure is order dependent.

Indeed, suppose that the output of the extensive-form rationalizability procedure would not depend on the order or speed of elimination. Then, we could use the following "backwards" order of elimination: First, eliminate all strategies that are suboptimal at non-terminal histories of degree 1, that is, at non-terminal histories that are only followed by terminal histories. Given the induced conditional belief vectors in round 1, then eliminate in round 2 those strategies that are suboptimal at non-terminal histories of degree 2, that is, at non-terminal histories that are only followed by terminal histories and non-terminal histories of degree 1. And so on. By doing so, we would generate a procedure that selects precisely the backward induction strategies in the game. If the output of the extensive-form rationalizability procedure were order independent, then the original procedure would yield the backward induction strategies as well, and Battigalli's theorem would immediately follow. However, we have seen that the output of the extensive-form rationalizability procedure is order dependent, and hence this kind of proof is not possible.

4 Choice Sequences

We have seen that the strategies that are finally selected by the extensive-form rationalizability procedure crucially depend on the order or speed of elimination we use. This obviously complicates any attempt to prove Battigalli's theorem. The approach we use in this paper is the following: In Section 5 we will present a recursive elimination procedure that characterizes, for every k, the *terminal histories* that are reachable under strategies that survive round k of the extensive-form rationalizability procedure. In particular, this procedure characterizes the extensive-form rationalizable outcomes in the game. We then show in Section 6 that this procedure is *order independent.* Moreover, we will see in Section 7 that there is a "backwards" order of elimination that finally leads to the backward induction outcome in the game. As the set of terminal histories selected by the procedure is order independent, the original procedure, which characterizes the extensive-form rationalizable outcomes, must also uniquely yield the backward induction outcome. Battigalli's theorem thus follows.

The main building block of the procedure will be the notion of a *choice sequence,* which is the sequence of choices for a player on the path from the root to some (terminal or non-terminal) history in the game. This notion is borrowed from von Stengel (1996), who introduced it to enable a computationally efficient analysis of large dynamic games. The procedure we present in Section 5 will recursively eliminate choice sequences, instead of strategies, from the game.

In this section we will first provide a formal definition of choice sequences, and then we will define what it means for a choice sequence to be *strictly dominated*. We will finally prove that a choice sequence is part of a rational strategy – that is, part of a strategy that is rational for some conditional belief vector – if and only if the choice sequence is not strictly dominated. The latter result will be key to our proof of Battigalli's theorem.

4.1 Definition

For a (terminal or non-terminal) history $h^* \in H \cup Z$ and a player i, let $\sigma_i[h^*] = (\sigma_i(h))_{h \in \hat{H}_i}$ be the collection of player i choices on the path from the root to h^*. That is, $\hat{H}_i$ is the set of histories in H_i that precede h^*, and for every $h \in \hat{H}_i$ the choice $\sigma_i(h)$ is the unique choice in $C_i(h)$ that leads to h^*.

Definition 4.1 (Choice sequence (von Stengel, 1996)) *A **choice sequence** for player i is a collection of choices $\sigma_i = (\sigma_i(h))_{h \in \hat{H}_i}$, where $\hat{H}_i \subseteq H_i$ and $\sigma_i(h) \in C_i(h)$ for all $h \in \hat{H}_i$, such that $\sigma_i = \sigma_i[h^*]$ for some $h^* \in H \cup Z$.*

Note that for every player i the choice sequence $\sigma_i[\emptyset]$ is empty, since there are no choices on the path to $\emptyset$. In that case, we denote the choice sequence by $\emptyset$. In the game of Figure 1, for instance, the choice sequences for player 1 are $\emptyset, a, b, (b, e)$ and (b, f), whereas the choice sequences for player 2 are $\emptyset, c, d, (d, g)$ and (d, h).

4.2 Strictly Dominated Choice Sequences

Consider a choice sequence $\sigma_i = (\sigma_i(h))_{h \in \hat{H}_i}$ for player i, and a history $h \in H$. By $Z(\sigma_i, h)$ we denote the set of terminal histories that can be reached if the game starts at h, and player i chooses according to σ_i. By

$$\overline{u}_i(\sigma_i, h) := \max_{z \in Z(\sigma_i, h)} u_i(z)$$

we denote the maximum utility that player i can obtain if the game starts at h, and player i chooses according to σ_i.

Fix a player i. We will now define, for every history $h \in H \cup Z$, the *max-min utility* $\underline{u}_i(h)$ for player i at h. We also define player i's *max-min choice* $\underline{c}_{ii}(h)$ at every $h \in H_i$, and for every player $j \neq i$ and history $h \in H_j$ we define the *punishment choice* $\underline{c}_{ij}(h)$ for player j at h, viewed from i's perspective. We will do so by induction on the degree of h.

Consider first a history h of degree 0, which means that h is a terminal history. Define

$$\underline{u}_i(h) := u_i(h).$$

Now, suppose that $k \geq 1$, and assume that $\underline{u}_i(h)$ has already been defined for all $h \in H \cup Z$ with degree $d(h) \leq k-1$. Take a history $h \in H$ of degree k. If $h \in H_i$, then let $\underline{c}_{ii}(h)$ be a choice at h with

$$\underline{u}_i(h\underline{c}_{ii}(h)) = \max_{c_i \in C_i(h)} \underline{u}_i(hc_i)$$

and let

$$\underline{u}_i(h) := \underline{u}_i(h\underline{c}_{ii}(h)).$$

If $h \in H_j$ with $j \neq i$, then let $\underline{c}_{ij}(h)$ be a choice at h with

$$\underline{u}_i(h\underline{c}_{ij}(h)) = \min_{c_j \in C_j(h)} \underline{u}_i(hc_j)\ ,$$

and let

$$\underline{u}_i(h) := \underline{u}_i(h\underline{c}_{ij}(h)).$$

In case there are several choices $\hat{c}_j$ with $\underline{u}_i(h\hat{c}_j) = \min_{c_j \in C_j(h)} \underline{u}_i(hc_j)$, then we choose $\underline{c}_{ij}(h)$ according to a fixed tie-breaking rule.

Since the game G is without relevant ties, the max-min choices $\underline{c}_{ii}(h)$ are unique for every $h \in H_i$. By induction on k, this procedure then defines the max-min utility $\underline{u}_i(h)$ for every $h \in H \cup Z$, the max-min choice $\underline{c}_{ii}(h)$ at every $h \in H_i$, and the punishment choice $\underline{c}_{ij}(h)$ for every player $j \neq i$ and every $h \in H_j$.

For every $h \in H$ and every player i, let $\underline{s}_{ii}[h]$ be the unique strategy for player i which at every $h' \in H_i$ preceding h selects the unique choice leading to h, and which at every other $h' \in H_i$ reachable under $\underline{s}_{ii}[h]$ selects the max-min choice $\underline{c}_{ii}(h)$. We call $\underline{s}_{ii}[h]$ the *max-min strategy* for player i at h.

Similarly, for every player $j \neq i$, let $\underline{s}_{ij}[h]$ be the unique strategy for player j which at every $h' \in H_j$ preceding h selects the unique choice leading to h, and which at every other $h' \in H_j$ reachable under $\underline{s}_{ij}[h]$ selects the punishment choice $\underline{c}_{ij}(h)$ from i's perspective. We call $\underline{s}_{ij}[h]$ the *punishment strategy*, from i's perspective, for player j at h.

It is well-known that for every $h \in H$ and every player i,

$$u_i(\underline{s}_{ii}[h], s_{-i}) \quad \geq \quad \underline{u}_i(h) \text{ for all } s_{-i} \in S_{-i}(h), \tag{4.1}$$

$$u_i(s_i, (\underline{s}_{ij}[h])_{j \neq i}) \quad \leq \quad \underline{u}_i(h) \text{ for all } s_i \in S_i(h), \tag{4.2}$$

and that

$$\underline{u}_i(h) = \max_{s_i \in S_i(h)} \min_{s_{-i} \in S_{-i}(h)} u_i(s_i, s_{-i}) = \min_{b_i(h) \in \Delta(S_{-i}(h))} \max_{s_i \in S_i(h)} u_i(s_i, b_i(h)). \tag{4.3}$$

Property (4.1) states that player i, by choosing his max-min strategy at h, can always guarantee his max-min utility at h. On the other hand, (4.2) states that player i cannot hope for more than his max-min utility at h if the opponents choose their punishment strategies. Property (4.3) is a consequence of (4.1) and (4.2). All these properties follow from Chapter 15 in von Neumann and Morgenstern (1953).

We are now fully equipped to define a *strictly dominated* choice sequence.

Definition 4.2 (Strictly dominated choice sequence) *A choice sequence* $\sigma_i = (\sigma_i(h))_{h \in \hat{H}_i}$ *is* ***strictly dominated*** *at history* $h \in \hat{H}_i$ *if* $\overline{u}_i(\sigma_i, h) < \underline{u}_i(h)$.

Recall that $\overline{u}_i(\sigma_i, h)$ is the highest utility that player i can achieve from h onwards if he chooses in accordance with σ_i. Moreover, by (4.1), player i can guarantee the utility $\underline{u}_i(h)$ at h by choosing in accordance with his max-min strategy at h. Hence, a choice sequence σ_i is strictly dominated at h precisely when every strategy in $S_i(h)$ that is consistent with σ_i is strictly dominated by i's max-min strategy at h. This explains the name "strictly dominated" choice sequence. We finally say that a choice sequence $\sigma_i = (\sigma_i(h))_{h \in \hat{H}_i}$ is strictly dominated if it so at some history $h \in \hat{H}_i$.

As an illustration, consider again the game G from Figure 1. The choice sequence (b, e) for player 1 is strictly dominated at $\emptyset$ since $\overline{u}_1((b, e), \emptyset) = 2$ and $\underline{u}_1(\emptyset) = 3$. For player 2, the choice sequence (d, h) is strictly dominated at h_3 since $\overline{u}_2((d, h), h_3) = 0$ and $\underline{u}_2(h_3) = 4$. In fact, (d, h) is also strictly dominated at h_1 since $\overline{u}_2((d, h), h_1) = 1$ and $\underline{u}_2(h_1) = 2$. It may be verified that all other choice sequences are not strictly dominated in this game.

4.3 Relation with Rational Strategies

We will now establish an important connection between undominated choice sequences and rational strategies. In Theorem 4.1 below we prove that a choice sequence is part of a rational strategy, if and only if, it is not strictly dominated. Here, we say that strategy s_i is *rational* if it is rational for some conditional belief vector $b_i = (b_i(h))_{h \in H_i}$ that satisfies Bayesian updating. Moreover, we say that a choice sequence $\sigma_i = (\sigma_i(h))_{h \in \hat{H}_i}$ is *part of a strategy* s_i if $\sigma_i(h) = s_i(h)$ for all $h \in \hat{H}_i$.

Theorem 4.1 (Undominated choice sequences versus rational strategies) *Let G be a finite dynamic game with perfect information and without relevant ties. Then, a choice sequence σ_i is part of a rational strategy, if and only if, σ_i is not strictly dominated.*

The proof can be found in the appendix. Hence, if we wish to verify whether a choice sequence can be extended to a rational strategy, it is sufficient to verify whether the choice sequence is strictly dominated or not. This result, which we believe is interesting in its own right, plays a central role in proving Battigalli's theorem.

We now give a brief sketch of the proof. Suppose first that $\sigma_i = (\sigma_i(h))_{h\in\hat{H}_i}$ is part of a rational strategy s_i. Then, at every $h \in \hat{H}_i$ the strategy s_i is optimal for some conditional belief $b_i(h)$. This implies that the expected utility $u_i(s_i, b_i(h))$ must be at least his max-min utility $\underline{u}_i(h)$. But then, the highest utility that player i can achieve at h by choosing in accordance with σ_i can only be larger, and hence will also be at least his max-min utility $\underline{u}_i(h)$. Therefore, σ_i is not strictly dominated at h.

Assume next that the choice sequence $\sigma_i = (\sigma_i(h))_{h\in\hat{H}_i}$ is not strictly dominated. Suppose player i believes, at every $h \in H_i$, (a) that his opponents will make the "rewarding" choices leading to $\overline{u}_i(\sigma_i, h)$ as long as he chooses in accordance with σ_i, and (b) that his opponents will make the punishment choices as soon as they discover that he has deviated from σ_i. Since σ_i is not strictly dominated, it will be optimal for player i to always choose in accordance with σ_i, and therefore σ_i can be extended to a strategy s_i that is rational for this conditional belief vector. The proof in the appendix is essentially a formalization of this argument.

On the basis of Theorem 4.1 we can easily characterize those outcomes that are reachable under rational strategies. Formally, we say that an outcome $z \in Z$ is *reachable under rational strategies* if there is a combination $(s_i)_{i\in I}$ of rational strategies that induces z.

Corollary 4.1 (Outcomes reachable under rational strategies) *Let G be a finite dynamic game with perfect information and without relevant ties. Then, an outcome $z \in Z$ is reachable under rational strategies, if and only if, for every player i the induced choice sequence $\sigma_i[z]$ is not strictly dominated.*

Proof. (a) Assume first that outcome z is reachable under rational strategies. Then, there is a combination of rational strategies $(s_i)_{i\in I}$ that induces z. For every player i, the induced choice sequence $\sigma_i[z]$ is then part of the rational strategy s_i. Hence, by Theorem 4.1, we conclude that for every player i, the choice sequence $\sigma_i[z]$ is not strictly dominated.

(b) Suppose that the outcome $z \in Z$ is such that for every player i, the induced choice sequence $\sigma_i[z]$ is not strictly dominated. By Theorem 4.1 it follows that for every player i, the choice sequence $\sigma_i[z]$ is part of a rational strategy s_i. But then, the combination $(s_i)_{i\in I}$ of rational strategies induces z, and hence z is reachable under rational strategies. ■

Hence, if we wish to verify whether a certain outcome can be reached if all players choose rationally, it is sufficient to check whether the induced choice sequences for the players are strictly dominated or not.

5 Iterated Elimination of Choice Sequences

In this section we will introduce an elimination procedure, called *iterated elimination of all strictly dominated choice sequences,* that iteratedly removes choice sequences and terminal histories from the game. Relying on Corollary 4.1 we can prove that this procedure characterizes, for every k, those terminal histories that are reachable under strategies that survive round k of the extensive-form rationalizability procedure. Consequently, the terminal histories that survive our procedure will be exactly the extensive-form rationalizable outcomes.

5.1 Definition

We first present our procedure called *iterated elimination of all strictly dominated choice sequences*. Before doing so, we need some additional notation.

Fix a finite dynamic game G with perfect information and without relevant ties. By $\Sigma_i(G)$ we denote the set of all choice sequences for player i in G, and by $\Sigma(G) := \cup_{i \in I} \Sigma_i(G)$ the set of all choice sequences of all players together. If $\hat{\Sigma} \subseteq \Sigma(G)$, we say that outcome $z \in Z$ is *reachable under* $\hat{\Sigma}$ if for every player i, and every $h \in H \cup Z$ weakly preceding z, it holds that $\sigma_i[h] \in \hat{\Sigma}$. By $Z(\hat{\Sigma})$ we denote the set of outcomes in G that are reachable under $\hat{\Sigma}$. For a set of terminal histories $\hat{Z} \subseteq Z$, let $G \cap \hat{Z}$ be the *reduced* game that is obtained if we restrict G to the terminal histories in $\hat{Z}$, and to the non-terminal histories and choices that precede $\hat{Z}$.

Definition 5.1 (Iterated elimination of all strictly dominated choice sequences) *Let G be a finite dynamic game with perfect information and without relevant ties. We recursively define sets of choice sequences Σ^k and reduced games G^k, as follows:*

(Initial step) *Set $\Sigma^0 := \Sigma(G)$ and $G^0 := G$.*

(Inductive step) *Let $k \geq 1$, and assume that G^{k-1} and Σ^{k-1} have already been defined. Then,*

$$\Sigma^k := \{\sigma \in \Sigma(G^{k-1}) \mid \sigma \text{ not strictly dominated in } G^{k-1}\},$$

and

$$G^k := G \cap Z(\Sigma^k).$$

A choice sequence $\sigma \in \Sigma(G)$ is said to survive iterated elimination of all strictly dominated choice sequences if $\sigma \in \Sigma^k$ for all k.

Note that at every round k, a choice sequence σ is either eliminated because it is no longer feasible, that is, $\sigma \notin \Sigma(G^{k-1})$, or because σ is strictly dominated in G^{k-1}. It is easily seen that for every k, the reduced game G^k is without relevant ties. By $\Sigma^\infty := \cap_{k \geq 0} \Sigma^k$ we denote the set of choice sequences that survive the iterated elimination of all strictly dominated choice sequences.

We illustrate the procedure by means of the game in Figure 1. Remember that the choice sequences for player 1 in G are $\emptyset, a, b, (b, e)$ and (b, f) and that the choice sequences for player 2 are $\emptyset, c, d, (d, g)$ and (d, h). Hence,

$$\Sigma^0 = \{\emptyset, a, b, (b, e), (b, f), c, d, (d, g), (d, h)\} \text{ and } G^0 = G.$$

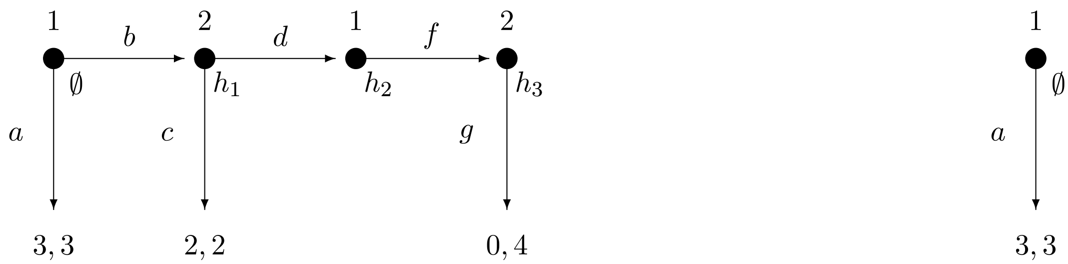

Figure 2: Reduced games G^1 and G^2

Also recall that the choice sequence (b,e) is strictly dominated for player 1 at $\emptyset$, that the choice sequence (d,h) is strictly dominated for player 2 at h_1 and h_3, and that all other choice sequences are not strictly dominated in G. Therefore, we eliminate the choice sequences (b,e) and (d,h) in round 1 and obtain

$$\Sigma^1 = \{\emptyset, a, b, (b,f), c, d, (d,g)\}.$$

By definition, $G^1 = G \cap Z(\Sigma^1)$. Note that the only terminal histories in $Z(\Sigma^1)$ are a, bc and $bdfg$. For instance, terminal history bde is not in $Z(\Sigma^1)$ since $\sigma_1[bde] = (b,e) \notin \Sigma^1$. Also, terminal history $bdfh$ is not in $Z(\Sigma^1)$ as $\sigma_2[bdfh] = (d,h) \notin \Sigma^1$. We thus conclude that

$$G^1 = G \cap \{a, bc, bdfg\},$$

which results in the game on the left-hand side of Figure 2.

Note that $\Sigma(G^1)$ contains the choice sequences $\emptyset, a, b$ and (b,f) for player 1, and contains the choice sequences $\emptyset, c, d$ and (d,g) for player 2. Hence,

$$\Sigma(G^1) = \{\emptyset, a, b, (b,f), c, d, (d,g)\}.$$

Within G^1, the choice sequences b and (b,f) are clearly strictly dominated for player 1 at $\emptyset$, since $\overline{u}_1(b, \emptyset, G^1) = \overline{u}_1((b,f), \emptyset, G^1) = 2$, whereas $\underline{u}_1(\emptyset, G^1) = 3$. Here, the entry G^1 in $\overline{u}_1(b, \emptyset, G^1)$ indicates that we are referring to the game G^1 (and not to G). Similarly for $\overline{u}_1((b,f), \emptyset, G^1)$ and $\underline{u}_1(\emptyset, G^1)$. In the sequel we will often add the game under consideration as an explicit entry in the different variables that we explore, so as to avoid any possible confusion.

Also, within G^1, the choice sequence c is strictly dominated for player 2 at h_1 since $\overline{u}_2(c, h_1, G^1) = 2$, whereas $\underline{u}_2(h_1, G^1) = 4$. We therefore eliminate the choice sequences $b, (b,f)$

and c from $\Sigma(G^1)$ and arrive at

$$\Sigma^2 = \{\emptyset, a, d, (d, g)\}.$$

By definition, $G^2 = G \cap Z(\Sigma^2)$. Note that the only terminal history in $Z(\Sigma^2)$ is a, which implies that

$$G^2 = G \cap \{a\}.$$

This game is depicted on the right-hand side of Figure 2.

We clearly have that

$$\Sigma(G^2) = \{\emptyset, a\},$$

since the choice sequences d and (d, g) for player 2 are no longer feasible in G^2. This implies immediately that

$$\Sigma^3 = \{\emptyset, a\},$$

after which the procedure terminates.

The iterated elimination of all strictly dominated choice sequences thus uniquely selects the terminal history a, which is the unique extensive-form rationalizable outcome of the game G.

5.2 Characterization of Extensive-Form Rationalizable Outcomes

Remember from Definition 3.1 that S_i^k is the set of strategies for player i that survive round k of the extensive-form rationalizability procedure. We say that an outcome $z \in Z$ is *reachable under* $(S_i^k)_{i \in I}$ if there is a strategy combination $(s_i)_{i \in I}$ in $\times_{i \in I} S_i^k$ that induces z. Remember also, from Definition 5.1, that G^k is the reduced game obtained at round k of the iterated elimination of all strictly dominated choice sequences. We now show that the iterated elimination of all strictly dominated choice sequences characterizes, for every k, the set of outcomes that are reachable under $(S_i^k)_{i \in I}$. As a consequence, it characterizes the extensive-form rationalizable outcomes in every game.

Theorem 5.1 (Characterization of extensive-form rationalizable outcomes) *Let G be a finite dynamic game with perfect information and without relevant ties. Then, for every $k \geq 0$, an outcome $z \in Z$ is reachable under $(S_i^k)_{i \in I}$, if and only if, z is a terminal history in G^k.*

The formal proof, which builds heavily on Corollary 4.1 and Theorem 3.1, can be found in the appendix. Here is the main idea of the proof.

Consider first round 1 of the extensive-form rationalizability procedure, which selects by definition the rational strategies for all players. Hence, the outcomes that a reachable under $(S_i^1)_{i \in I}$ are exactly the outcomes that are reachable under rational strategies, which – by Corollary 4.1 – are exactly the outcomes reachable under undominated choice sequences in G. These, in turn, are precisely the outcomes in G^1.

By definition, every conditional belief vector in B_i^1 strongly believes the event that the opponents choose rational strategies. That is, at every history $h \in H_i$ that is reachable under rational strategies, player i believes that his opponents indeed choose rational strategies. By our argument above, a history h is reachable under rational strategies precisely when it is reachable under undominated choice sequences, which is exactly the case when h is part of the reduced game G^1. Hence, a conditional belief vector in B_i^1 believes at every player i history in G^1 that his opponents will choose strategies that lead to terminal histories in G^1.

We also know, by Theorem 3.1, that in order to test for optimality of player i's strategy at a given history, it is sufficient to restrict to strategies in S_i^1. By Corollary 4.1, these are exactly the strategies for player i that lead to terminal histories in G^1. Altogether we thus see that at every history in G^1, the beliefs of the player in round 1 of the extensive-form rationalizability procedure, together with his own relevant strategies, can be described completely within the reduced game G^1.

Consider now round 2 of the extensive-form rationalizability procedure, which selects those strategies for player i that are rational for conditional belief vectors in B_i^1. Since we have seen above that, at every history in G^1, the conditional belief vectors in B_i^1 and the relevant strategies for player i can be described entirely within the reduced game G^1, we can apply the argument above to the game G^1 to conclude that the outcomes in G^1 that are reachable under $(S_i^2)_{i\in I}$ are exactly the outcomes in G^2. And so on. By continuing in this fashion we can prove this statement for every further round k as well. The proof in the appendix is essentially a formal implementation of this argument.

An immediate consequence of Theorem 5.1 is that the reduced game G^k is always nonempty. Indeed, we know that for every k the sets S_i^k are always nonempty, and therefore there is always at least one outcome z that is reachable under $(S_i^k)_{i\in I}$. But then, by the theorem above, there is for every k always at least one outcome z in G^k. Hence, the game G^k generated in round k of the iterated elimination of all strictly dominated choice sequences is always a "nonempty game".

6 Order Independence

In this section we will prove that the procedure of iterated elimination of all strictly dominated choice sequences is, in a specific sense, order independent. In order to explain what we mean by this "specific sense", let us call a choice sequence $\sigma_i = (\sigma_i(h))_{h\in\hat{H}_i}$ *regular* if at the last history h^* in $\hat{H}_i$ there is at least one other choice available besides $\sigma_i(h^*)$. The type of order independence we show is the following: If we first eliminate a *regular* strictly dominated choice sequence from the game, and then apply the procedure to the reduced game, then the output will be the same as when we would apply the procedure to the original game. That is, the output is insensitive to the elimination of an arbitrary regular strictly dominated choice sequence. By applying this order independence property repeatedly, it then follows that the output of the procedure remains unchanged if we successively eliminate regular strictly dominated choice sequences, in an arbitrary order, from the game.

In the next section we will show that this specific form of order independence is sufficient to prove Battigalli's theorem. As we will see, the backward induction procedure can be mimicked by the successive elimination of *regular* strictly dominated choice sequences, and hence the order independence property above implies that the backward induction procedure must yield the same outcome as the iterated elimination of all strictly dominated choice sequences. Together with Theorem 5.1, which guarantees that the latter procedure yields the extensive-form rationalizable outcomes, it follows that extensive-form rationalizability yields the same outcome as backward induction. That is, Battigalli's theorem follows.

In order to prove the order independence property above, we start by showing that a strictly dominated choice sequence will remain strictly dominated if we eliminate another *regular* strictly dominated choice sequence from the game. This monotonicity property is really the key to proving the order independence. The intuitive reason is that, if we would first remove a regular strictly dominated choice sequence from the game, and subsequently apply the procedure to the reduced game, then – by the monotonicity property – every strictly dominated choice sequence in the original game can still be removed in the reduced game because it remains strictly dominated there. Therefore, it should not matter for the final output whether we apply the procedure to the original game or to the reduced game. We show in this section that this informarmal argument can be turned into a formal proof for the order independence.

At this stage the reader may wonder why in this proof we focus on *regular,* and not on arbitrary, strictly dominated choice sequences. The reason is that the monotonicity property above does not hold if we remove a *non-regular* strictly dominated choice sequence from the game. To see this, consider the game G^1 in Figure 2. Note that the non-regular choice sequence (b, f) for player 1 is strictly dominated at $\emptyset$, and that the choice sequence c for player 2 is strictly dominated at h_1. If we eliminate the choice sequence (b, f) from the game G^1, the reduced game $G^1 \backslash (b, f)$ only contains the terminal histories a and bc. Hence, in the reduced game $G^1 \backslash (b, f)$, the choice sequence c is no longer strictly dominated for player 2, as c is the only choice for player 2 remaining in that game. Therefore, the monotonicity property fails here.

6.1 Monotonicity

Consider a choice sequence $\sigma_i = (\sigma_i(h_i))_{h \in \hat{H}_i}$ for player i in the game G, and let h^* be the last history in $\hat{H}_i$. We say that the choice sequence σ_i is *regular in* G if there are at least two available choices at h^* in G.

We will show in this subsection that the elimination of strictly dominated choice sequences is *monotonic* in the following sense: A choice sequence that is strictly dominated in G will *remain* strictly dominated if we eliminate from G a regular choice sequence that is strictly dominated. Formally, for a choice sequence $\sigma \in \Sigma(G)$ we denote by

$$G \backslash \sigma := G \cap Z(\Sigma(G) \backslash \{\sigma\})$$

the reduced game that results if we restrict to those histories that can be reached without σ. We informally refer to $G\backslash\sigma$ as the game that results if we "eliminate the choice sequence σ from the game".

We will show the monotonicity property by the following sequence of results: We first characterize the reduced game that remains after eliminating a regular choice sequence from the game. On the basis of this characterization, we then prove that the max-min utility of a player at a certain history can never decrease if we remove a regular strictly dominated choice sequence from the game. This result is then the key to proving the monotonicity property mentioned above.

We start by characterizing the reduced game that remains after eliminating a regular choice sequence from the game.

Lemma 6.1 (Game after eliminating a regular choice sequence) *Let G be a finite dynamic game with perfect information and without relevant ties, let $\sigma_i^* = (\sigma_i^*(h))_{h\in\hat{H}_i}$ be a regular choice sequence for player i in G, and let h^* be the last history in $\hat{H}_i$. Then, $G\backslash\sigma_i^*$ is the game obtained from G if we eliminate the choice $\sigma_i^*(h^*)$ and all histories that weakly follow $h^*\sigma_i^*(h^*)$).*

Proof . Let $\hat{G}$ be the the game obtained from G if we eliminate the choice $\sigma_i^*(h^*)$ and all histories that weakly follow $h^*\sigma_i^*(h^*)$). Note that $\hat{G}$ is a well-defined game as there is another available choice at h^*, and hence we do not eliminate all choices at h^*. We show that $Z(G\backslash\sigma_i^*) = Z(\hat{G})$, by proving the following two set inclusions: (a) $Z(G\backslash\sigma_i^*) \subseteq Z(\hat{G})$, and (b) $Z(\hat{G}) \subseteq Z(G\backslash\sigma_i^*)$.

(a) Take some $z \in Z(G\backslash\sigma_i^*) = Z(\Sigma(G)\backslash\{\sigma_i^*\})$. Then, for every $h \in H \cup Z$ weakly preceding z, we have that $\sigma_j[h] \in \Sigma(G)\backslash\{\sigma_i^*\}$ for every player j. In particular, it follows that $\sigma_i[h] \neq \sigma_i^*$ for every $h \in H \cup Z$ weakly preceding z. But then z cannot weakly follow $h^*\sigma_i^*(h^*)$, and hence $z \in Z(\hat{G})$.

(b) Take some $z \in Z(\hat{G})$. Then, z does not weakly follow $h^*\sigma_i(h^*)$, and hence $\sigma_i[h] \neq \sigma_i^*$ for every $h \in H \cup Z$ that weakly precedes z. Therefore we have, for every $h \in H \cup Z$ that weakly precedes z, that $\sigma_j[h] \in \Sigma(G)\backslash\{\sigma_i^*\}$ for every player j, which implies that $z \in Z(\Sigma(G)\backslash\{\sigma_i^*\}) = Z(G\backslash\sigma_i^*)$. ■

Hence, by the lemma above, the reduced game $G\backslash\sigma_i^*$ is obtained by eliminating the last choice in σ_i^*, together with all the histories that follow it, provided σ_i^* is regular. We use this result to prove that the max-min utility for a player at a given history can never decrease if we eliminate a regular strictly dominated choice sequence from the game.

Lemma 6.2 (Monotonicity of max-min utility) *Let G be a finite dynamic game with perfect information and without relevant ties, and let σ^* be a regular strictly dominated choice sequence in G. Then, for every $h \in H(G\backslash\sigma^*)$ and every player i, it holds that $\underline{u}_i(h, G\backslash\sigma^*) \geq \underline{u}_i(h, G)$.*

The proof can be found in the appendix. The main argument in the proof is the following. Consider first the case that σ^* belongs to a player $j \neq i$. Then, by Lemma 6.1, the game $G \backslash \sigma^*$ is obtained by eliminating the last choice for player j in σ^*, and all of its subsequent histories. Hence, i's opponents have less opportunity to "punish" player i, which implies that $\underline{u}_i(h, G \backslash \sigma^*) \geq \underline{u}_i(h, G)$.

Consider next the case that σ^* belongs to player i himself. Let $\sigma^* = (\sigma_i^*(h))_{h \in \hat{H}_i}$ and let h^* be the last history in σ^*. Then, by Lemma 6.1, the game $G \backslash \sigma^*$ is obtained by eliminating the choice $\sigma_i^*(h^*)$ at h^*, and all of its subsequent histories. It can then be shown that player i, by making the max-min choices in G everywhere, will certainly avoid the history h^*. Indeed, assume, on the contrary, that all choices in σ^* would be max-min choices in G. Then, at every history $h \in \hat{H}_i$, the max-min utility $\underline{u}_i(h, G)$ would be feasible under σ^*, and hence σ^* could not be strictly dominated in G, which is a contradiction. We thus conclude that player i, by making his max-min choices in G everywhere, will certainly avoid h^*, and will therefore guarantee that the play will stay within $G \backslash \sigma^*$. As such, player i's max-min strategy in G will also be feasible in $G \backslash \sigma^*$, and hence $\underline{u}_i(h, G \backslash \sigma^*) \geq \underline{u}_i(h, G)$.

On the basis of Lemma 6.2 we can easily establish the monotonicity property mentioned above.

Corollary 6.1 (Monotonicity) *Let G be a finite dynamic game with perfect information and without relevant ties, and let σ^* be a regular strictly dominated choice sequence in G. Then, every choice sequence in $\Sigma(G \backslash \sigma^*)$ that is strictly dominated in G is also strictly dominated in $G \backslash \sigma^*$.*

Proof. Suppose that the choice sequence $\sigma_i = (\sigma_i(h))_{h \in \hat{H}_i}$ for player i in $\Sigma(G \backslash \sigma^*)$ is strictly dominated in G. Then, there is some $h \in \hat{H}_i$ where

$$\overline{u}_i(\sigma_i, h, G) < \underline{u}_i(h, G).$$

Since it is clearly the case that $\overline{u}_i(\sigma_i, h, G \backslash \sigma^*) \leq \overline{u}_i(\sigma_i, h, G)$, and since $\underline{u}_i(h, G \backslash \sigma^*) \geq \underline{u}_i(h, G)$ by Lemma 6.2, it follows that

$$\overline{u}_i(\sigma_i, h, G \backslash \sigma^*) < \underline{u}_i(h, G \backslash \sigma^*),$$

and hence σ_i is strictly dominated in $G \backslash \sigma^*$. ■

We will now show that this monotonicity property still holds if we successively eliminate a *series* of regular strictly dominated choice sequences, instead of only one. Consider a reduced game $\hat{G} = G \cap Z(\hat{\Sigma})$ where $\hat{\Sigma} \subseteq \Sigma(G)$. We say that *$\hat{G}$ is reachable from G by iterated elimination of single regular strictly dominated choice sequences* if there are choice sequences $\sigma^1, ..., \sigma^M \in \Sigma(G)$ such that, for every $m \in \{1, ..., M\}$,

$$\sigma^m \in \Sigma((...(G \backslash \sigma^1) \backslash ...) \backslash \sigma^{m-1}), \tag{6.1}$$

$$\sigma^m \text{ is regular in } (...(G \backslash \sigma^1) \backslash ...) \backslash \sigma^{m-1} \text{ and strictly dominated in } G, \tag{6.2}$$

and such that

$$\hat{G} = (...(G \backslash \sigma^1) \backslash ...) \backslash \sigma^M. \tag{6.3}$$

We show, by means of Corollary 6.1, that every choice sequence in $\hat{G}$ that is strictly dominated in G is also strictly dominated in $\hat{G}$.

Lemma 6.3 (Monotonicity under iterated elimination) *Let G be a finite dynamic game with perfect information and without relevant ties, and let the reduced game $\hat{G} = G \cap Z(\hat{\Sigma})$ be reachable from G by iterated elimination of single regular strictly dominated choice sequences. Then, every choice sequence in $\hat{G}$ that is strictly dominated in G is also strictly dominated in $\hat{G}$.*

Proof. Let $\sigma^1, ..., \sigma^M \in \Sigma(G)$ be such that (6.1), (6.2) and (6.3) hold. Let $\sigma \in \Sigma(\hat{G})$ be a choice sequence that is strictly dominated in G. As, by (6.2), σ^1 is regular and strictly dominated in G, it follows by Corollary 6.1 that σ is strictly dominated in $G \backslash \sigma^1$ also.

Note that, by (6.1) and (6.2), $\sigma^2 \in \Sigma(G \backslash \sigma^1)$, σ^2 is strictly dominated in G and σ^1 is regular and strictly dominated in G. Hence, by Corollary 6.1, σ^2 is strictly dominated in $G \backslash \sigma_1$. Together with (6.2), we conclude that σ_2 is regular and strictly dominated in $G \backslash \sigma_1$. As we have seen that $\sigma \in \Sigma(\hat{G})$ is strictly dominated in $G \backslash \sigma^1$, it follows from Corollary 6.1 that σ is strictly dominated in $(G \backslash \sigma^1) \backslash \sigma^2$ also. By continuing in this fashion it follows that σ is strictly dominated in $(...(G \backslash \sigma^1) \backslash ...) \backslash \sigma^M = \hat{G}$. ■

6.2 Order Independence

In this subsection we will prove the order independence property described at the beginning of this section. That is, we will show that if we first eliminate an arbitrary regular strictly dominated choice sequence from the game, and then apply the procedure to the reduced game, we obtain the same output as when we would have applied the procedure to the original game.

We prove this order independence through the following sequence of results. We first show that if at a given history h of player i all choice sequences with the last choice at h are strictly dominated, then the shorter choice sequence $\sigma_i[h]$ must be strictly dominated as well. Recall that G^k is the reduced game at round k of the iterated elimination of all strictly dominated choice sequences, and that it is obtained from G^{k-1} by eliminating all strictly dominated choice sequences in G^{k-1}. Our second result, which heavily relies on the first, shows that if a reduced game $\hat{G}$ is obtained from G^{k-1} by the successive elimination of some regular strictly dominated choice sequences in G^{k-1}, then G^k is obtained from $\hat{G}$ by the successive elimination of some regular strictly dominated choice sequences in $\hat{G}$. On the basis of this result we then show the following nestedness property of the procedure: Suppose we eliminate a regular strictly dominated choice sequence σ from the game G, and that we compare the procedure applied to the original game G with the procedure applied to the reduced game $G \backslash \sigma$. Then, we show that the reduced game from round k of the first procedure always contains the reduced game from round k of the second procedure which, in turn, contains the reduced game from round $k+1$ of the first procedure. A consequence of this nestedness property is that both procedures must yield the same output, and hence the independence property described above follows.

We start by showing that if at a given history h of player i all choice sequences with the last choice at h are strictly dominated, then the shorter choice sequence $\sigma_i[h]$ must be strictly dominated as well.

Lemma 6.4 (If all choice sequences with the last choice at h are strictly dominated) *Let G be a finite dynamic game with perfect information and without relevant ties, and let $h \in H_i$. Suppose that, for every $c_i \in C_i(h)$, the choice sequence $\sigma_i[hc_i]$ is strictly dominated. Then, the reduced choice sequence $\sigma_i[h]$ is strictly dominated as well.*

Proof. Let z^* be a terminal history following h that yields the highest utility for player i among all terminal histories following h. Let $c_i^* \in C_i(h)$ be the choice at h that leads to z^*. Then, clearly, $\overline{u}_i(\sigma_i[hc_i^*], h) \geq \underline{u}_i(h)$, and hence $\sigma_i[hc_i^*]$ cannot be strictly dominated at h. Since, by assumption, $\sigma_i[hc_i^*]$ is strictly dominated, there must be some $h' \in H_i$ preceding h such that $\overline{u}_i(\sigma_i[hc_i^*], h') < \underline{u}_i(h')$. As $\overline{u}_i(\sigma_i[h], h') = \overline{u}_i(\sigma_i[hc_i^*], h')$, it follows that $\overline{u}_i(\sigma_i[h], h') < \underline{u}_i(h')$, and hence $\sigma_i[h]$ is strictly dominated. ■

Recall that G^k is the game obtained at round k of the iterated elimination of all strictly dominated choice sequences. That is, G^k is obtained from G^{k-1} by the elimination of *all* strictly dominated choice sequences in G^{k-1}. Consider now some reduced game $\hat{G}$ that is reachable from G^{k-1} by iterated elimination of *some,* but not necessarily all, *single regular* strictly dominated choice sequences in G^{k-1}. We show that G^k is then reachable from $\hat{G}$ by iterated elimination of single regular strictly dominated choice sequences in $\hat{G}$.

Lemma 6.5 (Zigzag Lemma) *Let G be a finite dynamic game with perfect information and without relevant ties. For some $k \geq 1$, consider a reduced game $\hat{G} = G \cap Z(\hat{\Sigma})$ with $\hat{\Sigma} \subseteq \Sigma(G)$ such that $\hat{G}$ is reachable from G^{k-1} by iterated elimination of single regular strictly dominated choice sequences. Then, G^k is reachable from $\hat{G}$ by iterated elimination of single regular strictly dominated choice sequences.*

The left-hand side of Figure 3 provides a visual representation of this result. Here, the arrow from G^{k-1} to $\hat{G}$ means that $\hat{G}$ is reachable from G^{k-1} by iterated elimination of single regular strictly dominated choice sequences. The same applies to the other arrow. The picture in Figure 3 explains the name "Zigzag Lemma".

The proof can be found in the appendix. Here is the main idea: Choose $\sigma^1, ..., \sigma^M$ to be the *minimal* choice sequences in $\hat{G}$ that are strictly dominated in G^{k-1}. Since $\hat{G}$ is is obtained from G^{k-1} by the elimination of *some* strictly dominated choice sequences in G^{k-1}, and G^k is

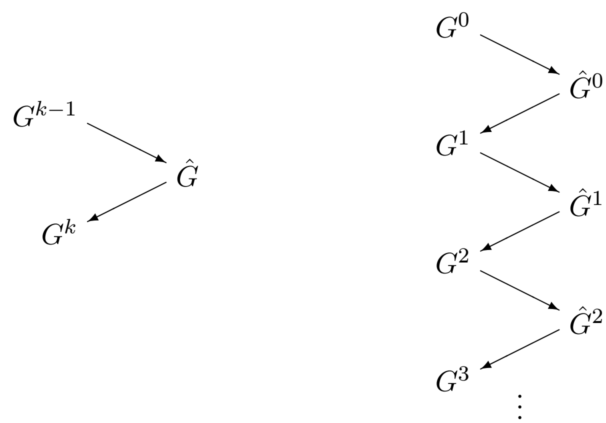

Figure 3: Visual representation of Lemma 6.5 and Lemma 6.6

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obtained from G^{k-1} by eliminating *all* choice sequences that are strictly dominated in G^{k-1}, it follows that

$$G^k = (...(\hat{G} \backslash \sigma^1) \backslash ...) \backslash \sigma^M.$$

Moreover, since $\sigma^1, ..., \sigma^M$ are *minimal* choice sequences in $\hat{G}$ that are strictly dominated in G^{k-1}, it can be shown on the basis of Lemma 6.4 that there is no history $h \in H$ such that $\{\sigma^1, ..., \sigma^M\}$ contains *all* choice sequences in $\Sigma(\hat{G})$ with the last choice at h. As a consequence, every choice sequence σ^m is regular in $(...(\hat{G} \backslash \sigma^1) \backslash ...) \backslash \sigma^{m-1}$.

To show that every σ^m is strictly dominated in $\hat{G}$, note that $\hat{G}$ is reachable from G^{k-1} by iterated elimination of single regular strictly dominated choice sequences. As σ^m is strictly dominated in G^{k-1}, it follows by Lemma 6.3 that σ^m is also strictly dominated in $\hat{G}$.

Overall, we thus see that $G^k = (...(\hat{G} \backslash \sigma^1) \backslash ...) \backslash \sigma^M$ is reachable from $\hat{G}$ by iterated elimination of single regular strictly dominated choice sequences.

Suppose now that we eliminate a regular strictly dominated choice sequence σ from the game G, and that we compare the procedure applied to the original game with the procedure applied to the reduced game $G \backslash \sigma$. On the basis of Lemma 6.5, we can show that both procedures are nested in the following sense.

Lemma 6.6 (Nestedness of procedures) *Let G be a finite dynamic game with perfect information and without relevant ties, and let σ be a regular strictly dominated choice sequence in G. Let $(\Sigma^k, G^k)_{k \geq 0}$ and $(\hat{\Sigma}^k, \hat{G}^k)_{k \geq 0}$ be the iterated elimination of all strictly dominated choice sequences when applied to the original game G and to the reduced game $G \backslash \sigma$, respectively. Then, for every $k \geq 0$,* *(a) $\hat{G}^k$ is reachable from G^k by iterated elimination of single regular strictly dominated choice sequences; and*

(b) G^{k+1} is reachable from $\hat{G}^k$ by iterated elimination of single regular strictly dominated choice sequences.

Proof. We prove statements (a) and (b) by induction on k.

For $k = 0$, (a) is clear since $\hat{G}^0 = G\backslash\sigma = G^0\backslash\sigma$, and σ is regular and strictly dominated in G.

To show (b) for $k = 0$, note that $\hat{G}^0$ is reachable from G^0 by iterated elimination of single regular strictly dominated choice sequences. Hence, by Lemma 6.5 we know that G^1 is reachable from $\hat{G}^0$ by iterated elimination of single regular strictly dominated choice sequences.

Suppose now that $k \geq 1$, and that (a) and (b) hold for $k - 1$.

To show (a) for k, note that, by the induction assumption on (b), G^k is reachable from $\hat{G}^{k-1}$ by iterated elimination of single regular strictly dominated choice sequences. Hence, it follows from Lemma 6.5 that $\hat{G}^k$ is reachable from G^k by iterated elimination of single regular strictly dominated choice sequences. Hence, (a) holds for k.

We now show that (b) holds for k. By (a) we know that $\hat{G}^k$ is reachable from G^k by iterated elimination of single regular strictly dominated choice sequences. Hence, it follows by Lemma 6.5 that G^{k+1} is reachable from $\hat{G}^k$ by iterated elimination of single regular strictly dominated choice sequences. Hence, (b) holds for k. This completes the proof. ■

Lemma 6.6 can be summarized visually by the the right-hand side of Figure 3. Here, the arrow from $\hat{G}^0$ to G^1 means that G^1 is reachable from $\hat{G}^0$ by iterated elimination of single regular strictly dominated choice sequences, and similarly for the other arrows. An immediate consequence of Lemma 6.6 is that the final outputs of both procedures must be the same. Indeed, suppose that both procedures terminate before round K. That is, $G^k = G^K$ and $\hat{G}^k = \hat{G}^K$ for all $k \geq K$. Then, it follows from Lemma 6.6 that $G^k = \hat{G}^k$ for all $k \geq K$. We thus obtain the following order independence property.

Theorem 6.1 (Order independence) *Let G be a finite dynamic game with perfect information and without relevant ties, and let σ be a regular strictly dominated choice sequence in G. Let $(\Sigma^k, G^k)_{k\geq 0}$ and $(\hat{\Sigma}^k, \hat{G}^k)_{k\geq 0}$ be the iterated elimination of all strictly dominated choice sequences when applied to the original game G and to the reduced game $G\backslash\sigma$, respectively. Then, there is some $K \geq 0$ such that $\hat{G}^k = G^k$ for all $k \geq K$.*

In the following subsection we will use this order independence property, together with Theorem 5.1, to prove Battigalli's theorem.

7 Proof of Battigalli's Theorem

With Theorems 5.1 and 6.1 at our disposal we are now ready to prove Battigalli's theorem.

Theorem 7.1 (Battigalli's theorem) *Let G be a finite dynamic game with perfect information and without relevant ties. Then, the only extensive-form rationalizable outcome in G is the backward induction outcome.*

Proof. Let $z^{bi} \in Z$ be the unique backward induction outcome in G, and let

$$G^{bi} := G \cap \{z^{bi}\}$$

be the reduced game that results if we restrict the game G to the single terminal history z^{bi} and its preceding non-terminal histories.

Now, let $(\Sigma^k, G^k)_{k\geq 0}$ be the iterated elimination of all strictly dominated choice sequences applied to the game G. By Theorem 5.1 we know that the output of this procedure characterizes exactly the extensive-form rationalizable outcomes in G. Hence, it is sufficient to show that there is some $K \geq 0$ such that $G^k = G^{bi}$ for all $k \geq K$.

To that purpose we will show that the backward induction procedure, leading to the game G^{bi}, can be mimicked by iterated elimination of single regular strictly dominated choice sequences from the game G.

We use the following notation. For a reduced game $\hat{G}$, let $H^*(\hat{G})$ be the set of last histories h in $H(\hat{G})$ where there are at least two available choices in $\hat{G}$. That is, there are at least two choices at h, and every non-terminal history following h contains only one choice.

Consider the sequence $(\hat{\Sigma}^k, \hat{G}^k)_{k\geq 0}$ of sets of choice sequences and reduced games given by

$$\hat{\Sigma}^0 := \Sigma(G) \text{ and } \hat{G}^0 := G,$$

and where, for all $k \geq 1$,

$$\hat{\Sigma}^k := \Sigma(\hat{G}^{k-1}) \backslash \{\sigma^k\} \tag{7.1}$$

for some σ^k in $\Sigma(\hat{G}^{k-1})$ such that (a) $\sigma^k = \sigma_i[hc_i]$ for some player i, some $h \in H^*(\hat{G}^{k-1}) \cap H_i(\hat{G}^{k-1})$, and some $c_i \in C_i(h, \hat{G}^{k-1})$ and (b) σ^k is strictly dominated at h. If $H^*(\hat{G}^{k-1})$ is empty, we simply set

$$\hat{\Sigma}^k := \Sigma(\hat{G}^{k-1}).$$

We then define

$$\hat{G}^k := G \cap Z(\hat{\Sigma}^k). \tag{7.2}$$

Note that (7.1), (7.2) and (a) and (b) above amount to eliminating, at every active round k, a last choice in the game $\hat{G}^{k-1}$ that is suboptimal, together with all histories that follow this choice. Since this is exactly what the backward induction procedure does, we conclude that there is some $K \geq 0$ such that

$$\hat{G}^k = G^{bi} \text{ for all } k \geq K. \tag{7.3}$$

Moreover, it is clear that every choice sequence σ^k above that is eliminated from the game $\hat{G}^{k-1}$ is, by construction, a regular choice in $\hat{G}^{k-1}$. Since the choice sequence σ^1 is regular and strictly dominated in G, we know from Theorem 6.1 that the iterated elimination of all strictly dominated choice sequences applied to the game G yields the same output as when we would apply the procedure to $\hat{G}^1 = G \backslash \sigma^1$. But then, since σ^2 is regular and strictly dominated in $\hat{G}^1$, it follows from Theorem 6.1 that the procedure applied to the game $\hat{G}^1$ yields the same output as when we would apply it to $\hat{G}^2 = \hat{G}^1 \backslash \sigma^2$. By continuing in this fashion, we conclude on the basis of (7.3) that applying the iterated elimination of all strictly dominated choice sequences to the game G yields the same output as when we would apply it to G^{bi}, which of course yields the outcome z^{bi}.

Together with Theorem 5.1 it follows that the unique extensive-form rationalizable outcome is z^{bi}, which completes the proof of Battigalli's theorem. ■

8 Concluding Remarks

8.1 Choice Sequences

The main methodological innovation of this paper is the use of choice sequences. Although the concept itself is not new – it has already been used in von Stengel (1996) for computational purposes – we believe this paper is the first to use it in the foundational branch of game theory. In the context of this paper, choice sequences turn out to be a powerful tool for dealing with *outcomes,* rather than strategies, in dynamic games. Based on this experience, we believe that choice sequences can also become important in other areas where the focus is on outcomes. Think, for instance, of implementation theory or mechanism design where the planner wishes to design a dynamic game in which a given concept leads to certain desirable *outcomes,* rather than strategies, in the game.

8.2 Reny's Theorem

Proposition 3 in Reny (1992) is, in terms of content and proof, very similar to Battigalli's theorem. It shows that in every dynamic game with perfect information and without relevant ties, the forward induction concept of *explicable equilibrium* yields a unique outcome: the backward induction outcome. Like Battigalli (1997), also Reny (1992) proves this result by using properties of fully stable sets (Kohlberg and Mertens (1986)). It would be interesting to see whether one can develop an alternative proof for this result similar to the proof for Battigalli's theorem in this paper.

8.3 Games with Imperfect Information

In this paper we have restricted our attention to dynamic games with perfect information. We believe, however, that the approach in this paper can be extended to games with *imperfect* information as well. For instance, it would be interesting to see whether one can extend the notion of a strictly dominated choice sequence in a meaningful way to games with imperfect information, such that an equivalent to Theorem 4.1 would hold. And if this is possible, whether one can use this result to develop a procedure, similar to the iterated elimination of all strictly dominated choice sequences in this paper, that characterizes the extensive-form rationalizable outcomes in *general* dynamic games. One could also look at other rationalizability concepts for dynamic games, like common belief in future rationality (Perea (2014)), and see whether one can develop a procedure that characterizes the outcomes induced by this concept. We leave these problems for future research.

8.4 Computational Efficiency

From a computational point of view, the advantage of using choice sequences rather than strategies is that the number of choice sequences is *linear* in the size of the game tree, whereas the number of strategies is *exponential*. Hence, if the procedure in this paper could be extended to one that characterizes the extensive-form rationalizable outcomes, or the outcomes induced by common belief in future rationality, in *general* dynamic games, this would be an important step forward for computational game theory. The reason is that such procedures would have a much lower complexity than the original procedures, since they use choice sequences rather than strategies.

9 Appendix

Proof of Theorem 3.1. The "only if" direction is trivially true, hence we only need to show the "if" direction. Suppose that s_i is such that there is some $b_i \in B_i^{k-1}$ with

$$u_i(s_i, b_i(h)) \geq u_i(s_i', b_i(h)) \text{ for all } s_i' \in S_i^{k-1} \cap S_i(h) \tag{9.1}$$

at all $h \in H_i$ that are reachable under s_i. We show that $s_i \in S_i^k$. Hence we must show that

$$u_i(s_i, b_i(h)) \geq u_i(s_i', b_i(h)) \text{ for all } s_i' \in S_i(h) \tag{9.2}$$

for all $h_i \in H_i$ that are reachable under s_i. We prove (9.2) by induction on the number of player i histories that precede h.

Suppose first that $h \in H_i$ is not preceded by any other $h' \in H_i$. Hence, $S_i(h) = S_i$. Assume, contrary to what we want to prove, that

$$u_i(s_i, b_i(h)) < u_i(s_i', b_i(h)) \text{ for some } s_i' \in S_i(h). \tag{9.3}$$

Since $b_i \in B_i^{k-1}$ satisfies Bayesian updating, it follows by Perea (2012, Lemma 8.14.1) that there is some $s_i'' \in S_i$ that is rational for b_i. Since $b_i \in B_i^{k-1}$, it follows that $s_i'' \in S_i^k$, and hence, in particular, $s_i'' \in S_i^{k-1}$. We thus have that $s_i'' \in S_i^{k-1} \cap S_i(h)$. Since s_i'' is rational for $b_i(h)$, it follows that

$$u_i(s_i'', b_i(h)) \geq u_i(s_i', b_i(h)). \tag{9.4}$$

By combining (9.3) and (9.4) we obtain that

$$u_i(s_i, b_i(h)) < u_i(s_i', b_i(h)) \leq u_i(s_i'', b_i(h)) \text{ with } s_i'' \in S_i^{k-1} \cap S_i(h),$$

which contradicts the assumption (9.1). Hence, we conclude that (9.2) must hold.

Suppose now that $h \in H_i$ is reachable under s_i, and that (9.2) holds for every $h' \in H_i$ that precedes h. We distinguish two cases.

(i) Assume first that there is some $h' \in H_i$ preceding h with $b_i(h')(S_{-i}(h)) > 0$. As b_i satisfies Bayesian updating and, by the induction assumption, s_i is rational for $b_i(h')$ at h', it follows by Perea (2012, Lemma 8.14.9) that s_i is rational for $b_i(h)$ at h. Hence, (9.2) holds at h.

(ii) Assume next that $b_i(h')(S_{-i}(h)) = 0$ for all $h' \in H_i$ preceding h. Suppose, contrary to what we want to prove, that

$$u_i(s_i, b_i(h)) < u_i(s_i', b_i(h)) \text{ for some } s_i' \in S_i(h). \tag{9.5}$$

Since b_i satisfies Bayesian updating, it follows by Perea (2012, Lemma 8.14.1) that there is some $s_i'' \in S_i$ that is rational for b_i. Let $\hat{H}_i$ be the set of histories $h'' \in H_i$ for which there is some $h' \in H_i$ preceding h with $b_i(h')(S_{-i}(h'')) > 0$.

Let s_i^* be the unique strategy such that

$$s_i^*(h') = \begin{cases} s_i(h'), & \text{if } h' \in \hat{H}_i \\ s_i''(h'), & \text{if } h' \notin \hat{H}_i \end{cases}$$

for all $h' \in H_i$ that are reachable under s_i^*. We show that s_i^* is rational for b_i.

Take some arbitrary $h' \in H_i$ that is reachable under s_i^*. We distinguish three cases.

(a) Assume first that h' precedes h. Then, $h' \in \hat{H}_i$ and s_i^* coincides with s_i on $\hat{H}_i$. Note that, by definition, all histories $h'' \in H_i$ with $b_i(h')(S_{-i}(h'')) > 0$ are in $\hat{H}_i$, and hence

$$u_i(s_i^*, b_i(h')) = u_i(s_i, b_i(h')).$$

Since, by the induction assumption, (9.2) holds at h', it follows that

$$u_i(s_i^*, b_i(h')) = u_i(s_i, b_i(h')) \geq u_i(s_i', b_i(h')) \text{ for all } s_i' \in S_i(h').$$

Hence, s_i^* is rational for $b_i(h')$ at h'.

(b) Assume next that h' does not precede h but $h' \in \hat{H}_i$. Then, by definition of $\hat{H}_i$, there is some $h'' \in H_i$ preceding h with $b_i(h'')(S_{-i}(h')) > 0$. We have seen in (a) that s_i^* is rational for $b_i(h'')$ at h''. Since b_i satisfies Bayesian updating, it follows from Perea (2012, Lemma 8.14.9) that s_i^* is rational for $b_i(h')$ at h'.

(c) Assume finally that $h' \notin \hat{H}_i$. Then, every $h'' \in H_i$ following h' will not be in $\hat{H}_i$ either. Hence, s_i^* coincides with s_i'' at h' and all $h'' \in H_i$ following h'. Therefore,

$$u_i(s_i^*, b_i(h')) = u_i(s_i'', b_i(h')).$$

Since, by construction, s_i'' is rational for b_i, it follows that s_i^* is rational for $b_i(h')$ at h'.

We thus conclude that s_i^* is rational for b_i. Since $b_i \in B_i^{k-1}$, it follows that $s_i^* \in S_i^k$, and hence, in particular, $s_i^* \in S_i^{k-1}$. Moreover, s_i^* coincides with s_i at $\hat{H}_i$, and hence coincides in particular with s_i at all player i histories preceding h. Therefore, $s_i^* \in S_i(h)$. We have thus constructed a strategy $s_i^* \in S_i^{k-1} \cap S_i(h)$ that is rational for b_i. In particular, s_i^* is rational for $b_i(h)$ at h, and hence

$$u_i(s_i^*, b_i(h)) \geq u_i(s_i', b_i(h)) \text{ for all } s_i' \in S_i(h). \tag{9.6}$$

Together with (9.5) this yields

$$u_i(s_i, b_i(h)) < u_i(s_i', b_i(h)) \leq u_i(s_i^*, b_i(h)) \text{ for some } s_i^* \in S_i^{k-1} \cap S_i(h),$$

which contradicts our assumption (9.1). Hence, we conclude that (9.2) must hold at h.

By induction, (9.2) will hold at every $h \in H_i$ that is reachable under s_i, which implies that s_i is rational for the conditional belief vector $b_i \in B_i^{k-1}$. Hence, we conclude that $s_i \in S_i^k$, which was to show. ■

Proof of Theorem 4.1. **(a)** Take an arbitrary choice sequence $\sigma_i = (\sigma_i(h))_{h \in \hat{H}_i}$ that is part of a rational strategy s_i. We show that σ_i is not strictly dominated.

Consider an arbitrary history $h \in \hat{H}_i$. As σ_i is part of the strategy s_i, it follows in particular that h is reachable under s_i. Since the strategy s_i is rational, there is a conditional belief $b_i(h) \in \Delta(S_{-i}(h))$ such that

$$u_i(s_i, b_i(h)) \geq u_i(s_i', b_i(h)) \text{ for all } s_i' \in S_i(h). \tag{9.7}$$

As σ_i is part of the strategy s_i, it follows that

$$\overline{u}_i(\sigma_i, h) \geq u_i(s_i, b_i(h)). \tag{9.8}$$

We also have, by (4.3) and (9.7), that

$$u_i(s_i, b_i(h)) = \max_{s_i' \in S_i(h)} u_i(s_i', b_i(h)) \ \geq \min_{b_i'(h) \in \Delta(S_{-i}(h))} \max_{s_i' \in S_i(h)} u_i(s_i', b_i'(h)) = \underline{u}_i(h). \tag{9.9}$$

By (9.8) and (9.9) it then follows that $\overline{u}_i(\sigma_i, h) \geq \underline{u}_i(h)$. Since this holds for every $h \in \hat{H}_i$, the choice sequence is not strictly dominated.

(b) Suppose now that the choice sequence $\sigma_i = (\sigma_i(h))_{h \in \hat{H}_i}$ is not strictly dominated. We show that σ_i is part of some rational strategy s_i.

We first introduce some new definitions. Remember that for every history $h \in H$, we denote by $Z(\sigma_i, h)$ the set of terminal histories that can be reached if the game starts at h and player i chooses according to σ_i. We have defined $\overline{u}_i(\sigma_i, h) := \max_{z \in Z(\sigma_i, h)} u_i(z)$ as the maximum utility that player i can achieve in $Z(\sigma_i, h)$. Let $\overline{z}_i(\sigma_i, h)$ be a terminal history in $Z(\sigma_i, h)$ such that $\overline{u}_i(\sigma_i, h) = u_i(\overline{z}_i(\sigma_i, h))$. If player j is active at h, then let $\overline{c}_{ij}(\sigma_i, h)$ be the unique choice in $C_j(h)$ that leads to $\overline{z}_i(\sigma_i, h)$. Moreover, for a given player i, we choose the terminal histories $\overline{z}_i(\sigma_i, h)$ in such a way that

$$\overline{z}_i(\sigma_i, h) = \overline{z}_i(\sigma_i, h') \text{ whenever } h \text{ follows } h' \text{ and } h \text{ precedes } \overline{z}_i(\sigma_i, h'). \tag{9.10}$$

Let s_i be the unique strategy such that

$$s_i(h) = \overline{c}_{ii}(\sigma_i, h) \tag{9.11}$$

for all $h \in H_i$ that is reachable under s_i.

We first show that σ_i is part of the strategy s_i. Take some $h \in \hat{H}_i$. Then, it must be that $\overline{c}_{ii}(\sigma_i, h) = \sigma_i(h)$ since $\overline{c}_{ii}(\sigma_i, h)$ is the choice at h that leads to the terminal history $\overline{z}_i(\sigma_i, h)$ which, by definition, weakly follows $h\sigma_i(h)$. This implies, by (9.11), that $s_i(h) = \sigma_i(h)$ for all $h \in \hat{H}_i$. Hence, σ_i is indeed part of s_i.

We will now construct a conditional belief vector $b_i = (b_i(h))_{h \in H_i}$ satisfying Bayesian updating for which s_i is optimal. The conditional belief vector b_i will be such that at every history $h \in H_i$, and for every opponent j, it assigns probability 1 to a strategy $s_j[h]$ which we will now define.

By $H(\sigma_i)$ we denote the set of non-terminal histories that are reachable under σ_i. Fix a history $h \in H_i$ and an opponent $j \neq i$. Remember that, for every $h' \in H_j$, we denote by $\underline{c}_{ij}(h')$ the punishment choice for player j at h', viewed from player i's perspective. Let $s_j[h]$ be the unique strategy in $S_j(h)$ such that

$$(s_j[h])(h') = \begin{cases} \overline{c}_{ij}(\sigma_i, h'), & \text{if } h' \in H(\sigma_i) \\ \underline{c}_{ij}(h'), & \text{otherwise} \end{cases} \tag{9.12}$$

for all $h' \in H_j$ that does not precede h and that is reachable under $s_j[h]$.

Let $b_i = (b_i(h))_{h \in H_i}$ be the conditional belief vector which at every $h \in H_i$ assigns probability 1 to the opponents' strategy combination $(s_j[h])_{j \neq i}$. Then, by construction, b_i satisfies Bayesian updating.

It remains to show that strategy s_i is rational for the conditional belief vector b_i. Fix a history $h \in H_i$ that is reachable under s_i. We will show that

$$u_i(s_i, b_i(h)) \geq u_i(s_i', b_i(h)) \text{ for all } s_i' \in S_i(h).$$

Suppose, contrary to what we want to prove, that $u_i(s_i, b_i(h)) < u_i(s_i', b_i(h))$ for some $s_i' \in S_i(h)$. Then, there is some $h' \in H_i$ weakly following h such that (i) h' is reachable under s_i and s_i', (ii) the choices $s_i(h')$ and $s_i'(h')$ are different at h', and (iii) $u_i(s_i, b_i(h')) < u_i(s_i', b_i(h'))$. Note that $h' \in H(\sigma_i)$, as h' is reachable under s_i and σ_i is part of s_i.

For every $h'' \in H_i$ that weakly follows h' and precedes $\overline{z}_i(\sigma_i, h')$ we have by (9.10) and (9.11) that

$$s_i(h'') = \overline{c}_{ii}(\sigma_i, h'') \text{ is the choice at } h'' \text{ leading to } \overline{z}_i(\sigma_i, h'). \tag{9.13}$$

Recall that $h' \in H(\sigma_i)$. Then, for every opponent j and every $h'' \in H_j$ that follows h' and precedes $\overline{z}_i(\sigma_i, h')$, it holds that $h'' \in H(\sigma_i)$, and hence we have by (9.10) and (9.12) that

$$(s_j[h'])(h'') = \overline{c}_{ij}(\sigma_i, h'') \text{ is the choice at } h'' \text{ leading to } \overline{z}_i(\sigma_i, h'). \tag{9.14}$$

By (9.13), (9.14) and the definition of $b_i(h')$ it then follows that

$$u_i(s_i, b_i(h')) = u_i(z(s_i, (s_j[h'])_{j \neq i})) = u_i(\overline{z}_i(\sigma_i, h')) = \overline{u}_i(\sigma_i, h'). \tag{9.15}$$

We now distinguish two cases: $h' \in \hat{H}_i$ and $h' \notin \hat{H}_i$, where $\hat{H}_i$ is the set of histories at which σ_i is defined.

Assume first that $h' \in \hat{H}_i$. Since, by the assumption in (ii), $s_i'(h') \neq s_i(h')$, and $s_i(h') = \sigma_i(h')$, it follows that $s_i'(h') \neq \sigma_i(h')$. Hence, the history $h's_i'(h')$, and all histories that follow, are not in $H(\sigma_i)$. By (9.12), the belief $b_i(h')$ then assigns probability 1 to the event that every opponent j will choose the punishment choice $\underline{c}_{ij}(h'')$ at every h'' weakly following $h's_i'(h')$. But then,

$$u_i(s_i', b_i(h')) = \underline{u}_i(h's_i(h')) \leq \underline{u}_i(h'). \tag{9.16}$$

By (9.15) and (9.16) and the assumption in (iii) that $u_i(s_i, b_i(h')) < u_i(s_i', b_i(h'))$, it follows that

$$\overline{u}_i(\sigma_i, h') = u_i(s_i, b_i(h')) < u_i(s_i', b_i(h')) \leq \underline{u}_i(h'),$$

which implies that

$$\overline{u}_i(\sigma_i, h') < \underline{u}_i(h').$$

This, however, contradicts our assumption that σ_i is not strictly dominated.

Assume next that $h' \notin \hat{H}_i$. Since $h' \in H(\sigma_i)$, as we have seen, and σ_i is not defined at h', nor at any $h'' \in H_i$ following h', we conclude that every $h'' \in H$ following h' will also be in $H(\sigma_i)$. Hence, by (9.12), the belief $b_i(h')$ assigns probability 1 to the event that every opponent j will choose the "rewarding" choice $\overline{c}_{ij}(\sigma_i, h'')$ at every h'' following h'. But then, the best possible strategy for player i at h' is to always choose $\bar{c}_{ii}(\sigma_i, h'')$ at every $h'' \in H_i$ following h', which, by (9.11), is exactly what s_i does. Hence, the assumption (iii) above that $u_i(s_i, b_i(h')) < u_i(s_i', b_i(h'))$ cannot be true.

We thus reach the conclusion that the assumption (iii), stating that $u_i(s_i, b_i(h')) < u_i(s_i', b_i(h'))$, cannot be true, and hence $u_i(s_i, b_i(h)) \geq u_i(s_i', b_i(h))$ for all $s_i' \in S_i(h)$. Since this holds for every $h \in H_i$ that is reachable under s_i, the strategy s_i is rational for the conditional belief vector b_i. As b_i satisfies Bayesian updating, we conclude that strategy s_i is rational. Since σ_i is part of s_i, we conclude that σ_i is part of a rational strategy, which was to show. ■

Proof of Theorem 5.1. For every k, let $Z(S^k)$ and $Z(G^k)$ be the sets of outcomes that are reachable under $(S_i^k)_{i \in I}$ and in G^k, respectively. Moreover, let $H(S^k)$ and $H(G^k)$ be the sets of non-terminal histories in G that are reachable under $(S_i^k)_{i \in I}$ and in G^k, respectively. Hence, we must show that $Z(S^k) = Z(G^k)$ for every k. We will prove this statement by induction on k.

For $k = 0$, the statement is trivially true, since $Z(S^0) = Z(G^0) = Z$.

Suppose now that $k \geq 1$, and assume that $Z(S^{k-1}) = Z(G^{k-1})$. We must show that $Z(S^k) = Z(G^k)$. Hence, we must prove two directions: (a) $Z(S^k) \subseteq Z(G^k)$; and (b) $Z(G^k) \subseteq Z(S^k)$.

(a) We show that $Z(S^k) \subseteq Z(G^k)$. Remember that $Z(\Sigma^k)$ is the set of outcomes reachable under Σ^k. Since $Z(G^k) = Z(\Sigma^k)$, it is sufficient to show that $Z(S^k) \subseteq Z(\Sigma^k)$. Take some $z \in Z(S^k)$. Then, there is some strategy combination $(s_i)_{i \in I}$ in $\times_{i \in I} S_i^k$ that induces z. We show that, for every player i and every $h \in H \cup Z$ weakly preceding z, the induced choice sequence $\sigma_i[h]$ is in Σ_i^k.

Fix a player i and some $h^* \in H \cup Z$ weakly preceding z, and let $\sigma_i[h^*] = (\sigma_i(h))_{h \in \hat{H}_i}$. In order to show that $\sigma_i[h^*] \in \Sigma_i^k$, we must show that $\sigma_i[h^*] \in \Sigma(G^{k-1})$, and that $\sigma_i[h^*]$ is not strictly dominated in G^{k-1}. As $z \in Z(S^k) \subseteq Z(S^{k-1})$ and, by the induction assumption, $Z(S^{k-1}) = Z(G^{k-1})$, it follows that $z \in Z(G^{k-1})$. Since h^* weakly precedes z, it follows that $\sigma_i[h^*] \in \Sigma(G^{k-1})$.

It remains to show that $\sigma_i[h^*] = (\sigma_i(h))_{h \in \hat{H}_i}$ is not strictly dominated in G^{k-1}. As z is induced by $(s_i)_{i \in I}$ in $(S_i^k)_{i \in I}$, and h^* weakly precedes z, it follows that $\sigma_i[h^*]$ is part of the strategy s_i. Since $s_i \in S_i^k$, strategy s_i is rational for some conditional belief vector $b_i \in B_i^{k-1}$.

Now, take some arbitrary $h \in \hat{H}_i$. Then, s_i is optimal for the belief $b_i(h)$ at h, hence

$$u_i(s_i, b_i(h)) \geq u_i(s_i', b_i(h)) \text{ for all } s_i' \in S_i^{k-1} \cap S_i(h). \tag{9.17}$$

Since $h \in \hat{H}_i$ and $\sigma_i[h^*]$ leads to z, the history h precedes the terminal history $z \in Z(S^{k-1})$. Hence, it follows that $h \in H(S^{k-1})$, which implies that $S_{-i}^{k-1} \cap S_{-i}(h) \neq \emptyset$. As $b_i \in B_i^{k-1}$, the conditional belief vector b_i strongly believes S_{-i}^{k-1}, and hence $b_i(h)(S_{-i}^{k-1}) = 1$. This means that

$$b_i(h) \in \Delta(S_{-i}^{k-1} \cap S_{-i}(h)). \tag{9.18}$$

By the induction assumption, $Z(S^{k-1}) = Z(G^{k-1})$. As a consequence, the set of terminal histories in G^{k-1} that follow h is

$$Z((S_j^{k-1} \cap S_j(h))_{j\in I}).$$

Since $s_i \in S_i^{k-1} \cap S_i(h)$, and $\sigma_i[h^*]$ is part of the strategy s_i, it immediately follows from (9.18) that

$$\overline{u}_i(\sigma_i[h^*], h, G^{k-1}) \geq u_i(s_i, b_i(h)). \tag{9.19}$$

We have seen above that the set of terminal histories in G^{k-1} that follow h is $Z((S_j^{k-1} \cap S_j(h))_{j\in I})$. Hence, by (4.3),

$$\underline{u}_i(h, G^{k-1}) = \min_{b_i'(h)\in\Delta(S_{-i}^{k-1}\cap S_{-i}(h))} \max_{s_i'\in S_i^{k-1}\cap S_i(h)} u_i(s_i', b_i'(h)). \tag{9.20}$$

By (9.17), (9.18) and (9.20) it then follows that

$$\begin{aligned} u_i(s_i, b_i(h)) &= \max_{s_i'\in S_i^{k-1}\cap S_i(h)} u_i(s_i', b_i(h)) \\ &\geq \min_{b_i'(h)\in\Delta(S_{-i}^{k-1}\cap S_{-i}(h))} \max_{s_i'\in S_i^{k-1}\cap S_i(h)} u_i(s_i', b_i'(h)) = \underline{u}_i(h, G^{k-1}). \end{aligned} \tag{9.21}$$

By (9.19) and (9.21) we may then conclude that

$$\overline{u}_i(\sigma_i[h^*], h, G^{k-1}) \geq \underline{u}_i(h, G^{k-1}),$$

which means that $\sigma_i[h^*]$ is not strictly dominated in G^{k-1} at h. Since this holds for every $h_i \in \hat{H}_i$, the choice sequence $\sigma_i[h^*]$ is not strictly dominated in G^{k-1}, and hence $\sigma_i[h^*] \in \Sigma^k$.

Since this holds for every player i, we conclude that $\sigma_i[h^*] \in \Sigma^k$ for every player i and every $h^* \in H \cup Z$ weakly preceding z, which implies that $z \in Z(\Sigma^k)$. As this holds for every $z \in Z(S^k)$, it follows that $Z(S^k) \subseteq Z(\Sigma^k) = Z(G^k)$, which was to show.

(b) We next show that $Z(G^k) \subseteq Z(S^k)$. Take some $z \in Z(G^k)$. Then, $z \in Z(\Sigma^k)$, as $Z(G^k) = Z(\Sigma^k)$. Hence, $\sigma_i[z] \in \Sigma^k$ for all players i. We show that $\sigma_i[z]$ is part of a strategy $s_i \in S_i^k$.

Fix a player i. Since $\sigma_i[z] \in \Sigma^k$ it holds, by definition, that $\sigma_i[z] \in \Sigma(G^{k-1})$ and that $\sigma_i[z]$ is not strictly dominated in G^{k-1}. By Theorem 4.1 we know that $\sigma_i[z]$ is part of a rational strategy $\tilde{s}_i$ in G^{k-1}.

Now, let $\hat{s}_i$ be some arbitrary strategy in S_i^k. Let s_i be the unique strategy in G such that

$$s_i(h) = \begin{cases} \tilde{s}_i(h), & \text{if } h \in H(G^{k-1}) \\ \hat{s}_i(h), & \text{if } h \notin H(G^{k-1}) \end{cases} \tag{9.22}$$

for all $h \in H_i$ that are reachable under s_i. Since $\sigma_i[z]$ is part of $\tilde{s}_i$, it immediately follows that $\sigma_i[z]$ is part of s_i as well.

We show that $s_i \in S_i^k$. To that purpose, we construct a conditional belief vector $b_i \in B_i^{k-1}$ for which s_i is rational. Consider an arbitrary history $h \in H_i$. We distinguish three cases.

(i) Assume first that $h \in H_i(G^{k-1})$, where $H_i(G^{k-1}) = H_i \cap H(G^{k-1})$. Since $\tilde{s}_i$ is rational in G^{k-1}, there is a conditional belief vector $\tilde{b}_i = (\tilde{b}_i(h'))_{h' \in H_i(G^{k-1})}$ in G^{k-1} satisfying Bayesian updating for which $\tilde{s}_i$ is rational. Suppose now that h is reachable under s_i. Then, $\tilde{s}_i$ is rational for the belief $\tilde{b}_i(h)$ at h in G^{k-1}. That is,

$$u_i(\tilde{s}_i, \tilde{b}_i(h)) \geq u_i(\tilde{s}_i', \tilde{b}_i(h)) \text{ for all } \tilde{s}_i' \in S_i(h, G^{k-1}). \tag{9.23}$$

By the induction assumption, $Z(S^{k-1}) = Z(G^{k-1})$. This implies that the set of terminal histories in G^{k-1} following h is

$$Z((S_j^{k-1} \cap S_j(h))_{j \in I}).$$

As $\tilde{s}_i$ coincides with s_i on $H(G^{k-1})$, it follows from (9.23) that there is some belief $b_i^*(h) \in \Delta(S_{-i}^{k-1} \cap S_{-i}(h))$, such that

$$u_i(\tilde{s}_i, \tilde{b}_i(h)) = u_i(s_i, b_i^*(h)) \geq u_i(s_i', b_i^*(h)) \text{ for all } s_i' \in S_i^{k-1} \cap S_i(h). \tag{9.24}$$

As $b_i^*(h) \in \Delta(S_{-i}^{k-1} \cap S_{-i}(h))$, the belief $b_i^*(h)$ is part of some conditional belief vector $b_i \in B_i^{k-1}$ in G. Hence, it follows from (9.24) and Theorem 3.1 that

$$u_i(s_i, b_i^*(h)) \geq u_i(s_i', b_i^*(h)) \text{ for all } s_i' \in S_i(h) \tag{9.25}$$

for all $h \in H_i(G^{k-1})$ that are reachable under s_i. Moreover, as $\tilde{b}_i$ satisfies Bayesian updating in G^{k-1}, we can choose the beliefs $(b_i^*(h))_{h \in H_i(G^{k-1})}$ in such a way that they satisfy Bayesian updating as well.

Let $\hat{H}_i(G^{k-1})$ be the set of histories $h \in H_i$ such that there is some $h' \in H_i(G^{k-1})$ preceding h with $b_i^*(h')(S_{-i}(h)) > 0$.

(ii) Assume next that $h \notin H_i(G^{k-1})$ but $h \in \hat{H}_i(G^{k-1})$. Then, by definition, there is some $h' \in H_i(G^{k-1})$ preceding h with $b_i^*(h')(S_{-i}(h)) > 0$. Let $b_i^{**}(h)$ be the Bayesian update of $b_i^*(h')$ at h, that is,

$$b_i^{**}(h)(s_{-i}) = \frac{b_i^*(h')(s_{-i})}{b_i^*(h')(S_{-i}(h))} \text{ for all } s_{-i} \in S_{-i}(h).$$

Since, by (9.25), s_i is rational for $b_i^*(h')$ at h', it follows by Perea (2012, Lemma 8.14.9) that s_i is rational for $b_i^{**}(h)$ at h whenever h is reachable under s_i. That is,

$$u_i(s_i, b_i^{**}(h)) \geq u_i(s_i', b_i^{**}(h)) \text{ for all } s_i' \in S_i(h) \tag{9.26}$$

for all $h \in \hat{H}_i(G^{k-1}) \backslash H_i(G^{k-1})$ that are reachable under s_i.

(iii) Assume finally that $h \notin H_i(G^{k-1})$ and $h \notin \hat{H}_i(G^{k-1})$. Then, all histories in H_i following h are also not in $H_i(G^{k-1})$ nor in $\hat{H}_i(G^{k-1})$. Hence, s_i coincides with $\hat{s}_i \in S_i^k$ at h and all $h' \in H_i$

following h. As $\hat{s}_i$ is in S_i^k, there is a conditional belief vector $\hat{b}_i \in B_i^{k-1}$, satisfying Bayesian updating, for which $\hat{s}_i$ is rational. In particular, $\hat{s}_i$ is optimal for $\hat{b}_i(h)$ at h, which means that

$$u_i(\hat{s}_i, \hat{b}_i(h)) \geq u_i(s_i', \hat{b}_i(h)) \text{ for all } s_i' \in S_i(h).$$

As s_i coincides with $\hat{s}_i$ at h and all $h' \in H_i$ following h, it follows that

$$u_i(\hat{s}_i, \hat{b}_i(h)) = u_i(s_i, \hat{b}_i(h)) \geq u_i(s_i', \hat{b}_i(h)) \text{ for all } s_i' \in S_i(h). \tag{9.27}$$

Now, define the conditional belief vector b_i by

$$b_i(h) := \begin{cases} b_i^*(h), & \text{if } h \in H_i(G^{k-1}) \\ b_i^{**}(h), & \text{if } h \notin H_i(G^{k-1}),\ h \in \hat{H}_i(G^{k-1}) \\ \hat{b}_i(h), & \text{if } h \notin H_i(G^{k-1}),\ h \notin \hat{H}_i(G^{k-1}) \end{cases} \tag{9.28}$$

Then, it follows from (9.25), (9.26) and (9.27) that strategy s_i is rational for b_i. Moreover, by construction, b_i satisfies Bayesian updating.

It remains to show that the conditional belief vector b_i is in B_i^{k-1}. Hence, we must show b_i strongly believes S_{-i}^m for all $m \leq k-1$.

Consider some $m \leq k-1$ and some $h \in H_i$ with $S_{-i}^m \cap S_{-i}(h) \neq \emptyset$. We must show that $b_i(h)(S_{-i}^m) = 1$. We distinguish three cases.

(i) Assume first that $h \in H_i(G^{k-1})$. Then, by construction, $b_i(h) = b_i^*(h) \in \Delta(S_{-i}^{k-1} \cap S_{-i}(h))$, which implies that $b_i(h)(S_{-i}^m) = 1$ since $m \leq k-1$.

(ii) Assume next that $h \in \hat{H}_i(G^{k-1}) \backslash H_i(G^{k-1})$. Then, by construction, $b_i(h) = b_i^{**}(h)$. Hence, there is some $h' \in H_i(G^{k-1})$ preceding h with $b_i^*(h')(S_{-i}(h)) > 0$, such that

$$b_i^{**}(h)(s_{-i}) = \frac{b_i^*(h')(s_{-i})}{b_i^*(h')(S_{-i}(h))} \text{ for all } s_{-i} \in S_{-i}(h). \tag{9.29}$$

We know from case (i) that $b_i^*(h')(S_{-i}^m) = 1$. But then, it follows from (9.29) that $b_i(h)(S_{-i}^m) = b_i^{**}(h)(S_{-i}^m) = 1$.

(iii) Assume finally that $h \notin H_i(G^{k-1})$ and $h \notin \hat{H}_i(G^{k-1})$. Then, by construction, $b_i(h) = \hat{b}_i(h)$. By assumption, $\hat{b}_i \in B_i^{k-1}$, and hence $\hat{b}_i$ strongly believes S_{-i}^m. Hence, $\hat{b}_i(S_{-i}^m) = 1$. Therefore, $b_i(h)(S_{-i}^m) = \hat{b}_i(h)(S_{-i}^m) = 1$.

We thus conclude that $b_i(h)(S_{-i}^m) = 1$ at all $h \in H_i$ where $S_{-i}^m \cap S_{-i}(h) \neq \emptyset$, and hence b_i strongly believes S_{-i}^m. Since this holds for every $m \leq k-1$, we conclude that $b_i \in B_i^{k-1}$.

Summarizing, we see that strategy s_i is rational for the conditional belief vector $b_i \in B_i^{k-1}$, which means, by definition, that $s_i \in S_i^k$. Since, for every player i, the choice sequence $\sigma_i[z]$

is part of s_i, it follows that $z \in Z(S^k)$. As this holds for every $z \in Z(G^k)$, we conclude that $Z(G^k) \subseteq Z(S^k)$, which was to show.

By combining (a) and (b), we have that $Z(S^k) = Z(G^k)$. By induction on k, we thus conclude that $Z(S^k) = Z(G^k)$ for every $k \geq 0$, which was to show. ■

Proof of Lemma 6.2. Suppose first that σ^* is a choice sequence for a player $j \neq i$. Let $\sigma^* = (\sigma_j^*(h'))_{h' \in \hat{H}_j}$ and let h^* be the last history in $\hat{H}_j$. Then, by Lemma 6.1, $G \backslash \sigma^*$ is obtained from G by eliminating the choice $\sigma_j^*(h^*)$ at h^* and all histories that weakly follow $h^* \sigma_j^*(h^*)$. Hence we have, by definition, that $\underline{u}_i(h^*, G \backslash \sigma^*) \geq \underline{u}_i(h^*, G)$. Moreover, it clearly holds that $\underline{u}_i(h', G \backslash \sigma^*) = \underline{u}_i(h', G)$ for all $h' \in H(G \backslash \sigma^*)$ that do not weakly precede $h^* \sigma_j^*(h^*)$. But then, it must be the case that $\underline{u}_i(h, G \backslash \sigma^*) \geq \underline{u}_i(h, G)$, as was to show.

We next consider the case that σ^* is a choice sequence for player i. Let $\sigma^* = (\sigma_i^*(h'))_{h' \in \hat{H}_i}$ and let h^* be the last history in $\hat{H}_i$. Again, by Lemma 6.1, $G \backslash \sigma^*$ is obtained from G by eliminating the choice $\sigma_i^*(h^*)$ at h^* and all histories that weakly follow $h^* \sigma_i^*(h^*)$. Since σ^* is, by assumption, strictly dominated, there must be some $h' \in \hat{H}_i$ such that σ^* is strictly dominated at h' in G. That is,

$$\overline{u}_i(\sigma^*, h', G) < \underline{u}_i(h', G). \tag{9.30}$$

Recall that, for every $\hat{h} \in H_i$, we denote by $\underline{c}_{ii}(\hat{h}, G)$ the max-min choice for player i at $\hat{h}$ in G.

Claim. There must be some $\hat{h} \in \hat{H}_i$ weakly following h' such that $\sigma_i^*(\hat{h}) \neq \underline{c}_{ii}(\hat{h}, G)$.

Proof of claim. Suppose, contrary to what we want to show, that $\sigma_i^*(\hat{h}) = \underline{c}_{ii}(\hat{h}, G)$ for all $\hat{h} \in \hat{H}_i$ weakly following h'. Then, σ^* is part of the max-min strategy $\underline{s}_{ii}[h', G]$ for player i at h' in G. But then, for every $s_{-i} \in S_{-i}(h', G)$ we have that

$$\overline{u}_i(\sigma^*, h', G) \geq u_i(\underline{s}_{ii}[h', G], s_{-i}) \geq \underline{u}_i(h', G),$$

where the last inequality follows from (4.1). Hence, we conclude that $\overline{u}_i(\sigma^*, h', G) \geq \underline{u}_i(h', G)$, which contradicts (9.30). Hence, it must be the case that $\sigma_i^*(\hat{h}) \neq \underline{c}_{ii}(\hat{h}, G)$ for some $\hat{h} \in \hat{H}_i$ weakly following h', which was to show. This completes the proof of the claim.

By the claim it follows that player i, by playing his max-min strategy $\underline{s}_{ii}[h, G]$ at h in G, guarantees that the history $h^* \sigma_i^*(h^*)$ will not be reached. This implies, by Lemma 6.1, that $\underline{s}_{ii}[h, G]$ is actually a strategy in $G \backslash \sigma^*$, that is, $\underline{s}_{ii}[h, G] \in S_i(G \backslash \sigma^*)$. Let s_{-i} be an arbitrary strategy combination in $S_{-i}(h, G \backslash \sigma^*)$, and $\hat{s}_{-i}$ a strategy combination in $S_{-i}(h, G)$ that coincides with s_{-i} at $H(G \backslash \sigma^*)$. Since strategy $\underline{s}_{ii}[h, G]$ guarantees that only histories in $H(G \backslash \sigma^*)$ can be reached, it follows that

$$u_i(\underline{s}_{ii}[h, G], s_{-i}) = u_i(\underline{s}_{ii}[h, G], \hat{s}_{-i}) \geq \underline{u}_i(h, G),$$

where the last inequality follows from (4.1). Since this holds for every $s_{-i} \in S_{-i}(h, G\backslash\sigma^*)$, it follows by (4.3) that

$$\begin{aligned}\underline{u}_i(h, G\backslash\sigma^*) &= \max_{s_i \in S_i(h,G\backslash\sigma^*)} \min_{s_{-i} \in S_{-i}(h,G\backslash\sigma^*)} u_i(s_i, s_{-i}) \\ &\geq \min_{s_{-i} \in S_{-i}(h,G\backslash\sigma^*)} u_i(\underline{s}_{ii}[h, G], s_{-i}) \geq \underline{u}_i(h, G).\end{aligned}$$

Hence, $\underline{u}_i(h, G\backslash\sigma^*) \geq \underline{u}_i(h, G)$, which was to show. ■

Proof of Lemma 6.5. We will construct choice sequences $\sigma^1, ..., \sigma^M \in \Sigma(\hat{G})$ such that, for every $m \in \{1, ..., M\}$,

$$\sigma^m \in \Sigma((...(\hat{G}\backslash\sigma^1)\backslash...)\backslash\sigma^{m-1}), \tag{9.31}$$

$$\sigma^m \text{ is regular in } (...(\hat{G}\backslash\sigma^1)\backslash...)\backslash\sigma^{m-1} \text{ and strictly dominated in } \hat{G}, \tag{9.32}$$

and such that

$$G^k = (...(\hat{G}\backslash\sigma^1)\backslash...)\backslash\sigma^M. \tag{9.33}$$

Let $\Sigma^* = \{\sigma^1, ..., \sigma^M\}$ be the *minimal* choice sequences in $\Sigma(\hat{G})$ that are strictly dominated in G^{k-1}. That is, every choice sequence in Σ^* is strictly dominated in G^{k-1}, and for every choice sequence $\sigma \in \Sigma^*$ with the last choice defined at $h \in H_i$, the reduced choice sequence $\sigma_i[h] \in \Sigma(\hat{G})$ is not strictly dominated in G^{k-1}. We will show that $\sigma^1, .., .\sigma^M$ satisfy (9.31), (9.32) and (9.33) above.

Recall that G^k is obtained from G^{k-1} by eliminating *all* strictly dominated choice sequences in G^{k-1}, and that $\hat{G}$ is obtained from G^{k-1} by the elimination of *some* strictly dominated choice sequences in G^{k-1}. Therefore, $\Sigma(G^k) \subseteq \Sigma(\hat{G})$, and hence it follows by construction that

$$G^k = \hat{G} \cap Z(\Sigma(\hat{G})\backslash\Sigma^*). \tag{9.34}$$

In order to prove (9.31), (9.32) and (9.33), we need to know more about the structure of the set Σ^*. For every $\sigma \in \Sigma^*$, let $h^*(\sigma)$ be the last history at which σ is defined, and let

$$H^* = \{h^*(\sigma) \mid \sigma \in \Sigma^*\}$$

be the collection of such last histories. Since Σ^* contains the *minimal* choice sequences in $\Sigma(\hat{G})$ that are strictly dominated in G^{k-1}, a history in H^* cannot follow another history in H^*.

Moreover, we can show that for every $h \in H^* \cap H_i$ there is some $c_i \in C_i(h, \hat{G})$ such that $\sigma_i[hc_i] \notin \Sigma^*$. Suppose, contrary to what we want to show, that $\sigma_i[hc_i] \in \Sigma^*$ for all $c_i \in C_i(h, \hat{G})$. Then, $\sigma_i[hc_i]$ is strictly dominated in G^{k-1} for all $c_i \in C_i(h, \hat{G})$. By assumption, $\hat{G}$ is reachable from G^{k-1} by iterated elimination of single regular strictly dominated choice sequences. That is, we obtain $\hat{G}$ from G^{k-1} by eliminating choice sequences that are strictly dominated in G^{k-1}.

Since $h \in H(\hat{G})$, this means that for every $c_i \in C_i(h, G^{k-1}) \backslash C_i(h, \hat{G})$, the choice sequence $\sigma_i[hc_i]$ is strictly dominated in G^{k-1}. But then, it follows that, for every $c_i \in C_i(h, G^{k-1})$, the choice sequence $\sigma_i[hc_i]$ is strictly dominated in G^{k-1}. However, by Lemma 6.4, the reduced choice sequence $\sigma_i[h] \in \Sigma(\hat{G})$ is then also strictly dominated in G^{k-1}. This contradicts, however, the fact that for every $c_i \in C_i(h, \hat{G})$, the choice sequence $\sigma_i[hc_i] \in \Sigma^*$ is a *minimal* choice sequence in $\Sigma(\hat{G})$ that is strictly dominated in G^{k-1}. Hence, we conclude that, for every player i and every $h \in H^* \cap H_i$, there is some $c_i \in C_i(h, \hat{G})$ such that $\sigma_i[hc_i] \notin \Sigma^*$. This implies that every $\sigma \in \Sigma^*$ is a regular choice sequence in the reduced game $\hat{G} \cap Z(\Sigma(\hat{G}) \backslash \Sigma^* \backslash \{\sigma\})$ obtained from $\hat{G}$ by eliminating all choice sequences in Σ^* except σ.

Based on the two observations above, it follows by Lemma 6.1 that the game $\hat{G} \cap Z(\Sigma(\hat{G}) \backslash \Sigma^*)$ is obtained by eliminating, for every player i and every $h \in H^* \cap H_i$, the choices $c_i \in C_i(h)$ for which $\sigma_i[hc_i] \in \Sigma^*$, together with all histories that follow.

Moreover, we conclude that for every $m \in \{1, ..., M\}$,

$$\sigma^m \in \Sigma((...(\hat{G} \backslash \sigma^1) \backslash ...) \backslash \sigma^{m-1}), \text{ and} \tag{9.35}$$

$$\sigma^m \text{ is a regular choice sequence in } \Sigma((...(\hat{G} \backslash \sigma^1) \backslash ...) \backslash \sigma^{m-1}). \tag{9.36}$$

It also follows from the observations above that

$$\hat{G} \cap Z(\Sigma(\hat{G}) \backslash \Sigma^*) = (...(\hat{G} \backslash \sigma^1) \backslash ...) \backslash \sigma^M.$$

Together with (9.34) above we conclude that

$$G^k = (...(\hat{G} \backslash \sigma^1) \backslash ...) \backslash \sigma^M. \tag{9.37}$$

It remains to show that, for every $m \in \{1, ..., M\}$, the choice sequence σ^m is strictly dominated in $\hat{G}$. By assumption, $\hat{G}$ is reachable from G^{k-1} by iterated elimination of single regular strictly dominated choice sequences. Since σ^m is strictly dominated in G^{k-1}, it follows from Lemma 6.3 that σ^m is also strictly dominated in $\hat{G}$.

Together with (9.35), (9.36) and (9.37), it follows that G^k is reachable from $\hat{G}$ by iterated elimination of single regular strictly dominated choice sequences. This completes the proof. ■

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