Local Prior Expected Utility: a basis for utility representations under uncertainty

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Abstract

Models of decision-making under ambiguity are widely used in economics. One stream of such models results from weakening the independence axiom in Anscombe and Aumann [1963]. We identify necessary assumptions on independence to represent the decision maker's preferences such that he acts as if he maximizes expected utility with respect to a possibly local prior. We call the resulting representation Local Prior Expected Utility, and show that the prior used to evaluate a certain act can be obtained by computing the gradient of some appropriately defined utility mapping. The numbers in the gradient, moreover, can naturally be interpreted as the subjective likelihoods the decision maker assigns to the various states. Building on this result we provide a unified approach to the representation results of Maximin Expected Utility and Choquet Expected Utility and characterize the respective sets of priors.

JEL: D80, D81

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1 Introduction

Ever since Ellsberg [1961] published his well-known paradox the matter of ambiguity received great interest in decision-making. Based on his paradox, many models have been conceived in an attempt to reconcile theory with the paradox. One way of avoiding the paradox is to relax

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the Independence Axiom of Anscombe and Aumann [1963] (AA). This approach yielded two of the most well-known models of ambiguity: Schmeidler's [1989] Choquet Expected Utility (CEU), and Gilboa and Schmeidler's [1989] Maximin Expected Utility (MEU). Under both approaches the decision maker acts as if he maximizes utility with respect to a set of priors. In this paper we identify necessary assumptions on the preference relation to obtain a representation under which the decision maker maximizes utility with respect to a set of priors, called Local Prior Expected Utility (LEU). Moreover, we show that the prior used to evaluate a certain act is equal to the gradient of some appropriately defined utility mapping. We argue that the equality is not a mere technicality but coherent with the qualitative interpretation of a probabilistic belief. Based on this result we provide a unified approach to MEU, CEU, and AA and characterize the respective sets of priors.

A preference relation $\succeq$ on acts that satisfies the standard axioms Weak Order, Monotonicity, Continuity, Risk Independence, and Non-Degeneracy can be represented by a functional ψ that maps vectors of statewise utilities to "overall" utilities. We show that Monotonicity of ψ guarantees the existence of its Gâteaux derivative almost everywhere relying on a deep mathematical theorem by Chabrillac and Crouzeix [1987]. Using only the Gâteaux derivative of ψ as an analytical tool, we prove characterization results corresponding to LEU, CEU, MEU, and AA. Thus, our proofs do not require an understanding of complicated mathematics and can be understood with basic knowledge of real analysis. Within our approach we clearly identify the structure of the set of priors needed in these environments.

As a basis for our approach, we identify an axiom called Independence of Certainty Equivalents that is weaker than Gilboa and Schmeidler's [1989] C-Independence and weaker than Schmeidler's [1989] Comonotonic Independence, but together with Weak Order, Monotonicity, Continuity, Risk Independence, Non-Degeneracy, and Uncertainty Aversion induces the same restrictions on the preference relation on acts as the Gilboa-Schmeidler axioms. We then show that without assuming Uncertainty Aversion we can represent a decision maker's preferences by taking expectations of an affine utility function u with respect to a (possibly) different prior for every act, which we call LEU. The prior used is equal to the gradient of ψ at the vector of utilities induced by the act if the Gâteaux derivative exists. In case the Gâteaux derivative does not exist at a particular vector, we can approximate it by the Gâteaux derivatives of nearby acts. This is possible since we show that the Gâteaux derivative exists almost everywhere, and the utility representation of the preference relation must be continuous. Thus non-smoothness of ψ does not affect our analysis.

Relaxing the Independence Axiom further by requiring invariance with respect to translations but not invariance to rescaling still results in a prior representation. However, the prior is no longer necessarily equal to the gradient of ψ at the vector of utilities induced by the act, but to the gradient at a possibly different vector. This result can be shown under Weak C-Independence in the sense of Maccheroni et al. [2006]. If on the other hand, the invariance of translations of utility profiles is relaxed, there is no representation as in the previous two cases since the gradient is not guaranteed to be a prior anymore.

The paper proceeds by introducing the fundamental axioms and necessary notation in Section 2. These axioms on preference relations over acts guarantee the existence of a function ψ on vectors of utilities, which represents the preference relation. Section 3 gives a formal definition of the Gâteaux derivative of ψ, shows that it exists almost everywhere and is equivalent to ψ's gradient if taken into the direction of the location. Using the existence of the Gâteaux derivative almost everywhere, we show in Section 4 that under Weak Order, Monotonicity, Continuity, Risk Independence, Non-Degeneracy, and Independence of Certainty Equivalents ψ is C-Additive and homogeneous, which is sufficient to prove the representation result for LEU. In Section 5 we use this framework to prove the representation results corresponding to MEU, CEU, and the AA representation by showing how additional assumptions on the preference relation translate into features of ψ and its Gâteaux derivative. Thereby, we characterize each set of axioms by regions of acts that use the same prior. Finally, Section 6 discusses the implications of further relaxations of the Independence Axiom and the connection to Ghirardato et al. [2004] who also characterize the set of priors under C-Independence.

2 A Utility Function for Acts

We take the classical Anscombe-Aumann setup [Anscombe and Aumann, 1963] as a starting point, denote by $\Omega := \{1, \ldots, n\}$ the finite set of states of nature and by X the finite set of outcomes. Then let $\Delta(X)$ be the set of all probability distributions over X. An act is a mapping $f : \Omega \to \Delta(X)$, and $\mathcal{F}$ is the set of all such acts. By $f(\omega)$ we denote the probability distribution over X induced by f in state ω, and by $f(\omega)(x)$ the probability of receiving alternative x in state ω. If we take any lottery $p \in \Delta(X)$ and write $\langle p \rangle$, we mean the constant act such that $\langle p \rangle(\omega) = p$ for all $\omega \in \Omega$. Moreover, we define mixtures of acts point wise, i.e. for each $\alpha \in [0, 1]$, $\alpha f + (1 - \alpha)g$ is the act that gives the prize $\alpha f(\omega) + (1 - \alpha)g(\omega)$ in state $\omega \in \Omega$. Finally, for some affine function $u : \Delta(X) \to \mathbb{R}$ we define $u \circ f := (u(f(\omega)))_{\omega \in \Omega}$ and let $\mathbb{1}_\omega$ be the vector for $\omega \in \Omega$ where $\mathbb{1}_\omega(\omega) = 1$ and $\mathbb{1}_\omega(\omega^{'}) = 0$ for all $\omega^{'} \in \Omega$ such that $\omega^{'} \neq \omega$. By $\mathbb{1}_\Omega$ we then denote the vector with $\mathbb{1}_\Omega(\omega) = 1$ for all $\omega \in \Omega$. The following five Axioms are the basis for most models in decision theory.

Axiom 1 (Weak Order)**.** *For all $f, g, h \in \mathcal{F}$ we have either $f \succeq g$ or $g \succeq f$, and if $f \succeq g$ and $g \succeq h$, then $f \succeq h$.*

Axiom 2 (Monotonicity)**.** *For any two acts $f, g \in \mathcal{F}$ with $\langle f(\omega) \rangle \succeq \langle g(\omega) \rangle$ for all $\omega \in \Omega$ we have $f \succeq g$.*

Axiom 3 (Continuity)**.** *For any acts $f, g, h \in \mathcal{F}$ such that $f \succ g \succ h$, there exists some $\alpha \in (0, 1)$ such that $g \sim \alpha f + (1 - \alpha)h$.*

Axiom 4 (Nondegeneracy)**.** *There exist $f, g \in \mathcal{F}$ such that $f \succ g$.*

Axiom 5 (Risk Independence)**.** *If $x, y, z \in \Delta(X)$ and $\lambda \in (0, 1]$ then $\langle x \rangle \succ \langle y \rangle$ implies $\lambda \langle x \rangle + (1 - \lambda)\langle z \rangle \succ \lambda \langle y \rangle + (1 - \lambda)\langle z \rangle$.*

Proposition 1 is a well-known result in the literature [e.g. Cerreia-Vioglio et al., 2011, Ok, 2007]. For the sake of completeness we reproduced the proof based on Ok [2007] in Appendix A1.

Proposition 1. *If the preference relation $\succeq$ satisfies Axioms 1 to 5, then there exists an affine function u on $\Delta(X)$ with $u(p) \in [-1, 1]$ for all $p \in \Delta(X)$, $u(p^*) = 1$ for some $p^* \in \Delta(X)$ and $u(p_*) = -1$ for some $p_* \in \Delta(X)$, a utility function U on $\mathcal{F}$ with $U(\langle p \rangle) = u(p)$ for all $p \in \Delta(X)$, and a continuous and monotone function $\psi : [-1, 1]^\Omega \to \mathbb{R}$ such that*

$$U(f) = \psi(u \circ f),$$

for all $f \in \mathcal{F}$. Moreover, for every act $f \in \mathcal{F}$ we can find a constant act $p \in \Delta(X)$ such that $f \sim \langle p \rangle$.

Note that it follows from the definition of ψ and u that $\psi(\mathbb{1}_\Omega) = \psi(u \circ \langle p^* \rangle) = U(\langle p^* \rangle) = 1$. Thus, we can evaluate each act $f \in \mathcal{F}$ by computing a vector of utilities $u \circ f$ using the utility function u on lotteries, which we can then evaluate using ψ, a monotone and continuous function. The function ψ simplifies the mathematical treatment since we can directly work with utility vectors rather than vectors of lotteries.

3 Gradient of ψ

In the following we will establish that the Gâteaux derivative, a generalized directional derivative, of ψ exists almost everywhere. This is due to the fact that ψ is monotonic, i.e. for any $F, G \in [-1,1]^\Omega$ we have $\psi(F) \le \psi(G)$ whenever $F(\omega) \le G(\omega)$ for all $\omega \in \Omega$. Formally, the Gâteaux derivative is defined as follows.

Definition 1 (Gâteaux derivative)**.** The function ψ is said to be Gâteaux differentiable at $F \in [-1,1]^\Omega$ if

1. $\psi^{'}(F,G) := \lim_{\alpha \downarrow 0} \frac{\psi(F+\alpha G)-\psi(F)}{\alpha}$, exists for all $G \in [-1,1]^\Omega$, and if
2. $\psi^{'}(F, \lambda G + \gamma H) = \lambda \psi^{'}(F,G) + \gamma \psi^{'}(F,H)$ for all $G, H \in [-1,1]^\Omega$ and all $\lambda, \gamma \in \mathbb{R}$ such that $\lambda G + \gamma H \in [-1,1]^\Omega$.

However, it is not guaranteed that the Gâteaux derivative exists everywhere. By combining Theorem 14 in Chabrillac and Crouzeix [1987] and the fact that Fréchet differentiability implies Gâteaux differentiability, we obtain the following Theorem.

Theorem 1. *Monotone functions on $\mathbb{R}^n$ are almost everywhere Gâteaux differentiable.*

It ensures the existence of derivatives of monotone functions almost everywhere, i.e. in a set of points with measure one. Since ψ is monotonic we are thereby ensured that its Gâteaux derivative exist almost everywhere and thus the following corollary is immediate.

Corollary 1. *The function $\psi : [-1,1]^\Omega \to \mathbb{R}$ is almost everywhere Gâteaux differentiable.*

The fact that ψ is not differentiable everywhere does not cause a problem. Later we will show how one can remedy this limitation. We proceed by defining the gradient of ψ. Recall that $\Omega = \{1, \dots, n\}$.

Definition 2. Let the gradient of ψ at $F \in [-1,1]^\Omega$ be

$$(\nabla \psi)(F) := \left(\psi^{'}(F, \mathbb{1}_1), \psi^{'}(F, \mathbb{1}_2), \dots, \psi^{'}(F, \mathbb{1}_n) \right),$$

where $\mathbb{1}_m \in [-1,1]^\Omega$ for all $m \in \Omega$.

Instead of taking the derivative in the direction of every dimension, the Gâteaux derivative describes changes in multiple directions at the same time. Lemma 1 establishes the exact relation between the Gâteaux derivative and the gradient in our setting.

Lemma 1. *If the Gâteaux derivative of ψ at $F \in [-1,1]^\Omega$ exists, it holds that $\nabla \psi(F) G = \psi^{'}(F,G)$ for any $G \in [-1,1]^\Omega$.*

Hence, taking the gradient of ψ at some F and computing the vector product with some G is equivalent to taking the Gâteaux derivative at F in the direction of G, where $F, G \in [-1,1]^\Omega$. This gives us a simple way to go back and forth between the gradient and the Gâteaux derivative at some point. As we will see shortly this connection provides a simple way of not only proving a new representation result but also to prove the classical results in the ambiguity literature.

4 Representation for Local Prior Expected Utility

In order to obtain our representation result, we need to assume that the convex combination of an act $f \in \mathcal{F}$ with a constant act $\langle q \rangle$ is equivalent to the convex combination of f's certainty equivalent $\langle p \rangle$ with the same constant act $\langle q \rangle$, which we formalize in the following axiom.

Axiom 6 (Independence of Certainty Equivalent)**.** *If $f \in \mathcal{F}$ and $p, q \in \Delta(X)$ such that $f \sim \langle p \rangle$, then $\lambda f + (1-\lambda)\langle q \rangle \sim \lambda \langle p \rangle + (1-\lambda)\langle q \rangle$ for all $\lambda \in (0,1]$.*

From now on we assume that $\succeq$ also satisfies Axiom 6. This additional assumption gives rise to the following representation theorem.

Theorem 2. *Let $\succeq$ be a preference relation on $\mathcal{F}$ satisfying Axioms 1 to 6, represented by the functions ψ and u as described in Proposition 1, and suppose the Gâteaux derivative of ψ at $F \in [-1,1]^\Omega$ exists, where $F = u \circ f$ for some $f \in \mathcal{F}$, then*

$$U(f) = \nabla\psi(F)F.$$

Moreover, $\nabla\psi(F)$ is a probability measure.

Hence, a decision maker satisfying Axioms 1 to 6 behaves as if he evaluates an act $f \in \mathcal{F}$ by computing the expected utility of f given the respective prior $\nabla\psi(F)$ and the utility function u. Therefore, our representation result is similar to the classical Anscombe-Aumann representation. Only instead of a single prior, the decision maker uses a possibly different prior for every act. Moreover, as in the Anscombe-Aumann setup, the prior is equal to the gradient at the respective act, with the difference that in their setup the gradient is constant over all acts.

That the gradient is considered to be a prior is not only due to the fact that it satisfies the characteristics of a probability measure. The interpretation of changes in the overall utility due to a small local deviation in the direction of a particular state is tightly connected to the meaning of a probability distribution. Since the gradient describes reactions in overall utility to small changes in the utility of a certain state, the value of the gradient must describe how likely the decision maker deems that particular state. Thus, the value of the gradient also qualitatively identifies the likelihood of states according to the decision maker's preference.

To prove the representation theorem we will make heavy use of two properties of ψ called *positive homogeneity* and *C-Additivity*, which we establish here.

Proposition 2. *If the preference relation $\succeq$ satisfies Axioms 1 to 6, then ψ is*

1. *positively homogeneous, i.e. it holds that $\psi(\lambda F) = \lambda\psi(F)$ for all $\lambda > 0$ and all $F \in [-1,1]^\Omega$ such that $\lambda F \in [-1,1]^\Omega$, and it is*

2. *C-Additive, i.e. it holds that $\psi(F + \alpha \mathbb{1}_\Omega) = \psi(F) + \alpha\psi(\mathbb{1}_\Omega)$ for all $F \in [-1,1]^\Omega$ and $\alpha \geq 0$ such that $F + \alpha\mathbb{1}_\Omega \in [-1,1]^\Omega$.*

The existence of the Gâteaux derivative almost everywhere is such a strong result that we do not need to rely on any other advanced theorems. Positive homogeneity and C-Additivity of ψ together with the definition of the Gâteaux derivative are sufficient to prove the representation theorem. Lemma 2 is an important step as it establishes the connection between the utility of an act $f \in \mathcal{F}$ and the Gâteaux derivative at F in the direction of F if the Gâteaux derivative at F exists and $F = u \circ f$.

Lemma 2. *If the preference relation $\succeq$ on $\mathcal{F}$ satisfies Axioms 1 to 6, and the Gâteaux derivative of ψ at F exists, it holds that $\psi(F) = \psi'(F, F)$ for any $F \in [-1, 1]^\Omega$.*

When we combine Lemma 2 with Lemma 1, we can conclude that if the Gâteaux derivative of ψ exists at $F \in [-1, 1]^\Omega$, then $\psi(F) = \nabla\psi(F)F$. Thus, we are left with the task to show that $\nabla\psi(F)$ is a prior, which we establish in Proposition 3.

Proposition 3. *If the preference relation $\succeq$ on $\mathcal{F}$ satisfies Axioms 1 to 6 and the Gâteaux derivative of ψ at $F \in [-1, 1]^\Omega$ exists, the gradient $\nabla\psi(F)$ of ψ has the following properties:*

- $\nabla\psi(F)(\omega) \geq 0$ *for all* $\omega \in \Omega$, *and*
- $\sum_{\omega\in\Omega} \nabla\psi(F)(\omega) = 1$.

These are all the ingredients needed to prove Theorem 2. We show the formal proof here as it nicely illustrates how the previous results establish Theorem 2.

Proof of Theorem 2. It follows directly from Proposition 3 that for any $F \in [-1, 1]^\Omega$ the gradient $\nabla\psi(F)$ is a probability measure. By Lemma 1 and 2 we can now compute $\psi(F)$ using $\nabla\psi(F)F$. Since we defined ψ such that $U(f) = \psi(F)$, it follows that $U(f) = \nabla\psi(F)F$. □

In Theorem 2 we have shown that Axioms 1 to 6 imply the Local Prior Expected Utility representation as given in the statement of that theorem. We will now provide, by means of Theorem 3, a counterpart to this result by showing that every Local Prior Expected Utility representation with certain properties implies the Axioms 1 to 6.

Theorem 3. *The preference relation $\succeq$ satisfies Axioms 1 to 6 if and only if there exists*

- *an affine function $u : \Delta(X) \to \mathbb{R}$ with $u(p) \in [-1, 1]$ for all $p \in \Delta(X)$, and with $u(p^*) = 1$ for some $p^* \in \Delta(X)$ and $u(p_*) = -1$ for some $p_* \in \Delta(X)$,*
- *a utility function $U : \mathcal{F} \to \mathbb{R}$ representing $\succeq$ such that $U(\langle p\rangle) = u(p)$ for all $p \in \Delta(X)$, and such that for every $f \in \mathcal{F}$ there is a $q \in \Delta(X)$ with $U(f) = U(\langle q\rangle)$, and*
- *a continuous, monotone, positively homogeneous, and C-Additive function $\psi : [-1, 1]^\Omega \to \mathbb{R}$ such that $U(f) = \psi(u \circ f) = \nabla\psi(F)F$ for every $f \in \mathcal{F}$ and $F = u \circ f$.*

Finally, we discuss the situation when the Gâteaux derivative at $F \in [-1, 1]^\Omega$ does not exist. By continuity of ψ we can simply approximate the utility at such a point. Thus, from here on we assume that whenever the derivative for any particular act does not exist, we approximate it as suggested in Proposition 4.

Proposition 4. *If the preference relation $\succeq$ on $\mathcal{F}$ satisfies Axioms 1 to 6 and the Gâteaux derivative at $F \in [-1, 1]^\Omega$ does not exist, then there is a sequence $F_n \to F$ such that the Gâteaux derivative exists at every F_n, and such that $\psi(F) = \lim_{n\to\infty} \nabla\psi(F_n)F_n$.*

5 Representation for Classical Models of Decision Making Under Uncertainty

5.1 Maximin Expected Utility

Gilboa and Schmeidler [1989] relax the Full-Independence axiom of Anscombe and Aumann [1963] and replace it with a much weaker Axiom called C-Independence.

Axiom 7 (C-Independence)**.** *For any $f, g \in \mathcal{F}$, $p \in \Delta(X)$, and any $\lambda \in (0,1]$, $f \succeq g$ if and only if $\lambda f + (1-\lambda)\langle p\rangle \succeq \lambda g + (1-\lambda)\langle p\rangle$.*

It is a weakening of the Full-Independence Axiom in the sense that it only requires the independence property with respect to mixtures with constant acts. However, it turns out that we can prove Gilboa and Schmeidler's [1989] representation result with a weaker set of axioms. We do not need to assume Axiom 7 but can simply continue to work with Axioms 5 and 6. To see why those are weaker note that C-Independence implies both Axiom 5 and Axiom 6. Firstly, if we take two constant acts $\langle f\rangle, \langle g\rangle \in \mathcal{F}$ then Axiom 7 directly implies Axiom 5. Secondly, if we take any $f \in \mathcal{F}$ and its certainty equivalent $\langle p\rangle$ then Axiom 7 implies Axiom 6. Furthermore note that a preference relation that satisfies Axioms 1 to 6 is C-Additive and positively homogeneous. Using these properties it can be shown that such a preference relation also satisfies Axiom 7. The reason we chose to work with Axioms 5 and 6 instead of Axiom 7 is that this approach pinpoints the exact properties we need to prove the representation theorem.

The C-Independence Axiom was not the only novelty in Gilboa and Schmeidler's approach. They also introduced an axiom called Uncertainty Aversion, which implies that if the decision maker is indifferent between two acts then he will never prefer one of them to a mixture of the two acts.

Axiom 8 (Uncertainty Aversion)**.** *For any $f, g \in \mathcal{F}$, if $f \sim g$ then $\frac{1}{2}f + \frac{1}{2}g \succeq f$.*

Under the assumption that $\succeq$ satisfies Axioms 1 to 6 and Axiom 8 we can now prove their famous representation theorem. Note that our proof is again very simple in nature as it only exploits the definition of the Gâteaux derivative. In order to state their representation theorem we define $\mathcal{P}_G := \left\{\nabla\psi(F) : \text{ Gâteaux derivative of } \psi \text{ at } F \in [-1,1]^\Omega \text{ exists}\right\}$ and $\mathcal{P} := \overline{co}\,(\mathcal{P}_G)$, where $\overline{co}$ denotes the convex closure.

Theorem 4. *A preference relation $\succeq$ that satisfies Axioms 1 to 6 and Axiom 8 can be represented by*

$$U(f) := \min_{p \in \mathcal{P}} pF,$$

for any $f \in \mathcal{F}$ and $F = u \circ f$.

The theorem states that a decision maker with a preference relation $\succeq$, which satisfies Axioms 1 to 6 and Axiom 8, acts as if he evaluates each act $f \in \mathcal{F}$ using an affine utility function u on $\Delta(X)$ and the prior $p \in \mathcal{P}$ that induces the minimal expected utility for act f. The fact that the decision maker uses the prior that induces the minimal expected utility clearly stems from Axiom 8. Without Axiom 8 the decision maker evaluates each act with its respective prior, the gradient at that act, as shown in Theorem 2. Technically Axiom 8 causes ψ to be Super Additive, which we establish in the following proposition.

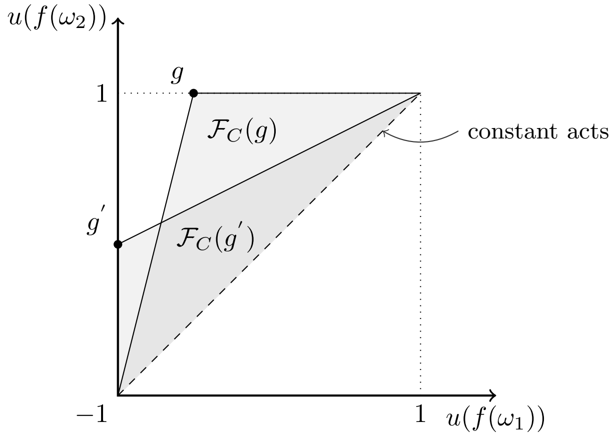

Figure 1: Acts represented as utility vectors for two states.

Proposition 5 (Super Additive)**.** *Suppose that the preference relation $\succeq$ on $\mathcal{F}$ satisfies Axioms 1 to 6 and Axiom 8. Let $\succeq$ be represented by ψ and u, as in Proposition 1, then for any $F, G \in [-1,1]^\Omega$ with $F + G \in [-1,1]^\Omega$ we have that $\psi(F+G) \geq \psi(F) + \psi(G)$.*[1]

The proof of Theorem 4 shows directly how Super Additivity of ψ lets the decision maker evaluate acts using the prior that induces the minimal expected utility.

Proof of Theorem 4. Take $f \in \mathcal{F}$ and let $F := u \circ f$, then we have to show that $U(f) \leq \nabla\psi(G)F$ for all $\nabla\psi(G) \in \mathcal{P}_G$. This is sufficient since Theorem 2 still holds and we therefore know that $U(f) = \nabla\psi(F)F$ and that $\nabla\psi(G)$ is a probability measure for all $G \in [-1,1]^\Omega$ whenever the Gâteaux derivative exists a G. By Lemma 1 it follows that $\nabla\psi(G)F = \psi^{'}(G,F)$. Using the definition of the Gâteaux derivative

$$\begin{aligned}
\psi^{'}(G,F) &= \lim_{\alpha\downarrow 0} \frac{\psi(G+\alpha F) - \psi(G)}{\alpha} \\
&\geq \lim_{\alpha\downarrow 0} \frac{\psi(G) + \psi(\alpha F) - \psi(G)}{\alpha} \\
&= \lim_{\alpha\downarrow 0} \frac{\alpha\psi(F)}{\alpha} \\
&= \psi(F),
\end{aligned}$$

where the inequality follows by Proposition 5, and the second equation by positive homogeneity of ψ. By definition $\psi(F) = U(f)$, and thus $U(f) \leq \nabla\psi(G)F$. Therefore we can evaluate every act using the prior $p \in \mathcal{P}$ that induces the minimum utility for act f. □

It is natural to ask how the set of priors in Theorem 4 is structured. To answer this question, we define the set of all acts in the same C-independent region as $g \in \mathcal{F}$ by $\mathcal{F}_C(g) := \{\lambda g + (1-\lambda)\langle p\rangle : \lambda \in (0,1] \text{ and } p \in \Delta(X)\}$. Note that our definition of $\mathcal{F}_C(g)$ does not include

[1] Proposition 5 is well-known. We reproduce the proof of Ok [2007] for the sake of completeness.

constant acts. This fact, however, does not limit the scope of our description as the utility of constant acts is independent of the prior. The following Proposition shows that each set $\mathcal{F}_C(g)$ can be described by a single prior $p \in \mathcal{P}$.

Proposition 6. *For an act $g \in \mathcal{F}$ and all $f \in \mathcal{F}_C(g)$ it holds that $U(f) = \nabla\psi(G)F$, where $F = u \circ f$ and $G = u \circ g$.*

Proof. Fix some $g \in \mathcal{F}$ and take any $f \in \mathcal{F}_C(g)$. Thus we must have for some $\lambda \in (0, 1]$ and some $p \in \Delta(X)$ that $f = \lambda g + (1-\lambda)\langle p\rangle$. Let $u(p) = \beta$. Then by affinity of u we have

$$\begin{aligned} F &= u \circ (\lambda g + (1-\lambda)\langle p\rangle) \\ &= \lambda u \circ g + (1-\lambda) u \circ \langle p\rangle \\ &= \lambda G + (1-\lambda)\beta \mathbb{1}_\Omega. \end{aligned}$$

Then,

$$\begin{aligned} \nabla\psi(G)F &= \psi^{'}(G, F) \\ &= \lim_{\alpha \downarrow 0} \frac{\psi(G + \alpha F) - \psi(G)}{\alpha} \\ &= \lim_{\alpha \downarrow 0} \frac{\psi(G + \alpha(\lambda G + (1-\lambda)\beta \mathbb{1}_\Omega) - \psi(G)}{\alpha} \\ &= \lim_{\alpha \downarrow 0} \frac{\psi((1+\alpha\lambda)G + \alpha(1-\lambda)\beta \mathbb{1}_\Omega) - \psi(G)}{\alpha} \\ &= \lim_{\alpha \downarrow 0} \frac{(1+\alpha\lambda)\psi(G) + \alpha(1-\lambda)\beta - \psi(G)}{\alpha} \\ &= \lim_{\alpha \downarrow 0} \frac{\alpha\lambda\psi(G) + \alpha(1-\lambda)\beta}{\alpha} \\ &= \lambda\psi(G) + (1-\lambda)\beta \\ &= \psi(\lambda G + (1-\lambda)\beta \mathbb{1}_\Omega) \\ &= \psi(F) \\ &= U(f), \end{aligned}$$

where the fifth and the eight equality hold by C-Additivity and positive homogeneity of ψ. □

To understand how the set of priors $\mathcal{P}$ looks like, observe that there could be $g, g^{'} \in \mathcal{F}$ with $g \neq g^{'}$ such that $\mathcal{F}_C(g) \cap \mathcal{F}_C(g^{'}) \neq \emptyset$ as depicted in Figure 1.[2] Thus we define the set $\mathcal{G} := \{g \in \mathcal{F} : g \notin \mathcal{F}_C(g^{'})$ for any $g^{'} \neq g\}$ as the set of acts that induce the "largest" C-independent region. For the two-state case we have $\mathcal{G} = \{(-1, 1), (1, -1)\}$ and therefore we only need two priors to describe the decision makers' preferences. Figure 2 gives an example of a $\mathcal{F}_C(g)$ with $g \in \mathcal{G}$ for the three-state case. Thus the set of priors can be exactly described by $\mathcal{P}_G = \{\nabla\psi(G) : G = u \circ g$ with $g \in \mathcal{G}\}$. Since the set $\mathcal{G}$ is possibly infinite when there are more than 2 states, the set $\mathcal{P}_G$ could be as well.

[2]For a matter of presentation we simplified the axes' labels. Complete labels would read: $u(f(\omega_i))$, $u(g(\omega_i))$, and $u(g^{'}(\omega_i))$ for state $i \in \Omega$.

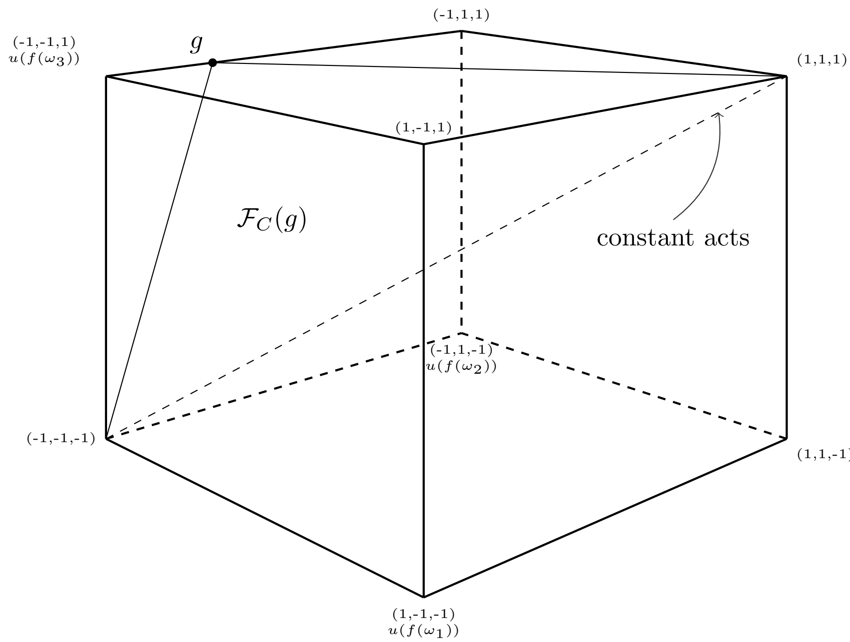

Figure 2: Acts represented as utility vectors for three states.

5.2 Choquet Expected Utility

Schmeidler's [1989] Choquet Expected Utility also relaxes the Independence Axiom of Anscombe-Aumann. Schmeidler introduces the notion of Comonotonicity and requires independence only for acts that are pairwise comonotonic. The idea behind Comonotonicity is that two comonotonic acts induce the same ranking over states.

Definition 3 (Comonotonicity)**.** Two acts $f, g \in \mathcal{F}$ are comonotonic if for no $\omega, \omega^{'} \in \Omega$, it holds that $\langle f(\omega)\rangle \succ \langle f(\omega^{'})\rangle$ and $\langle g(\omega^{'})\rangle \succ \langle g(\omega)\rangle$.

Building on the definition of Comonotonicity we can introduce Schmeidler's axiom of Comonotonic Independence.

Axiom 9 (Comonotonic Independence)**.** *For all pairwise comonotonic acts $f, g, h \in \mathcal{F}$ and for all $\lambda \in (0, 1)$, if $f \succ g$ then $\lambda f + (1-\lambda)h \succ \lambda g + (1-\lambda)h$.*

Comonotonic Independence is stronger than Independence of Certainty Equivalent and Risk Independence. However, we cannot compare it with C-Independence because Comonotonic Independence requires all three acts to be pairwise comonotonic. C-Independence, on the other hand, only requires the mixing act to be constant and thus Comonotonic Independence does not imply C-Independence. To see why Comonotonic Independence implies Risk Independence, take all three acts to be constant acts.[3] Since these three constant acts are pairwise comonotonic, Comonotonic

[3]Proposition 20 in Appendix A4 establishes that given Axioms 1 to 4 and Axiom 9 it holds that for all pairwise comonotonic acts $f, g, h \in \mathcal{F}$ such that $f \sim g$ we have $\lambda f + (1-\lambda)h \sim \lambda g + (1-\lambda)h$ where $\lambda \in (0, 1)$.

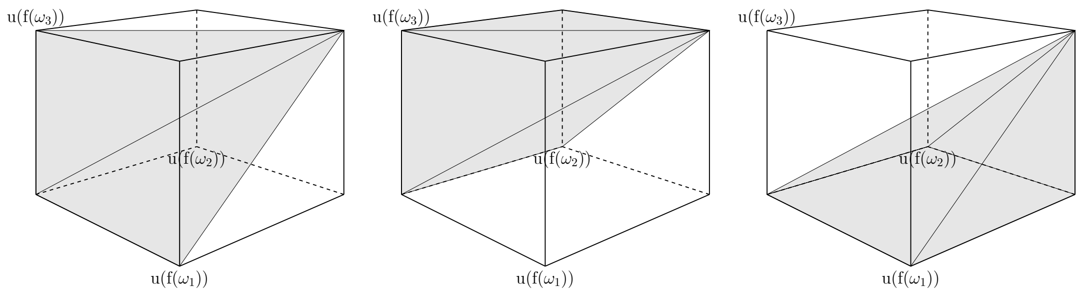

Figure 3: Example of comonotonic regions for three states.

Independence implies Risk Independence. For Independence of Certainty Equivalents we take an act, its certainty equivalent, and a constant act. Again, all acts are pairwise comonotonic because the certainty equivalent is also a constant act. Thus we can conclude that Comonotonic Independence implies Independence of Certainty Equivalents, which also illustrates why Comonotonic Independence leads to a stronger additivity result.

Proposition 7 (Comonotonic Additivity)**.** *Assume that $\succeq$ satisfies Axioms 1 to 4 and Axiom 9. Let $\succeq$ be represented by ψ and u as defined in Proposition 1, and take any two comonotonic acts $f, g \in \mathcal{F}$ and $F, G \in [-1,1]^\Omega$ such that $u \circ f = F$ and $u \circ g = G$. Then it holds that $\psi(F + \alpha G) = \psi(F) + \alpha\psi(G)$ for any $\alpha > 0$ such that $F + \alpha G \in [-1,1]^\Omega$.*

Compared to C-Additivity, Comonotonic Additivity also holds for two comonotonic acts and not only for an act and a constant act. It is immediate that Comonotonic Additivity implies C-Additivity. To discuss Choquet Expected Utility we need to introduce some further notation and the notion of a capacity. A capacity is a set valued function that assigns a number between 0 and 1 to every subset of Ω. Furthermore, it must hold that a subset cannot receive a higher value than its superset.

Definition 4 (Capacity)**.** A function $v : 2^\Omega \to \mathbb{R}$ is a capacity if it satisfies the following two conditions:

- $v(\emptyset) = 0$ and $v(\Omega) = 1$, and
- for all $E, G \in 2^\Omega$, we have that $E \subset G$ implies $v(E) \leq v(G)$.

Moreover, recall that $n := |\Omega|$, fix an $F \in [-1,1]^\Omega$ and let $\sigma : \{1, ..., n\} \to \Omega$ be a bijective function such that we have $F(\sigma(1)) \geq F(\sigma(2)) \geq \cdots \geq F(\sigma(n))$. Moreover, we define $F(\sigma(n+1)) = 0$. Then the Choquet integral of F with respect to the capacity v is defined as

$$\int_\Omega F dv = \sum_{i=1}^{n} \left[F(\sigma(i)) - F(\sigma(i+1))\right] v\left(\{\sigma(1), \ldots, \sigma(i)\}\right). \tag{1}$$

The idea behind Choquet Integration is to compute the area under a functions' graph by placing rectangles within the graph of the function. However, this is not done vertically as with Riemann integration but horizontally. An often used analogy is that of pouring water, i.e. filling up the graph from the bottom to the top with the value of the capacity as basis.[4]

[4]A more elaborate explanation can be found in Gilboa [2009].

These are all the ingredients needed to state and prove Schmeidler's representation theorem for Choquet Expected Utility.

Theorem 5. *For a preference relation $\succeq$ that satisfies Axioms 1 to 4 and Axiom 9 there exists a unique capacity v on Ω and an affine real valued function u on $\Delta(X)$ such that*

$$U(f) = \int_{\Omega} (u \circ f) dv, \tag{2}$$

for any $f \in \mathcal{F}$.

So if a decision maker satisfies Axioms 1 to 4 and Axiom 9 he acts as if he calculates the expected utility of an act using a capacity instead of a prior as in the classical Anscombe-Aumann setup. To prove Theorem 5 we define a capacity $v : 2^{\Omega} \to \mathbb{R}$ by $v(X) := \psi(\mathbb{1}_X)$ for every $X \subseteq \Omega$, where $\mathbb{1}_X$ is the vector in $[-1, 1]^{\Omega}$ with 1's where the respective state is included in $X \subseteq \Omega$ and 0's otherwise. We then show that computing the Choquet Integral with respect to v is equivalent to the representation obtained in Theorem 2. This approach is motivated by the idea that a capacity can be interpreted as a set of $n!$ priors–a relationship that is well-known in the literature. Therefore, we define the set of all acts in the same comonotonic region as $g \in \mathcal{F}$, by $\mathcal{F}_{Co}(g) := \{f \in \mathcal{F} : f \text{ is comonotonic to } g\}$. Proposition 8 establishes that we only need one prior for each possible comonotonic region. Moreover, there are as many comonotonic regions as there are strict ordering of the states in Ω.

Proposition 8. *Assume that $\succeq$ satisfies Axioms 1 to 4 and Axiom 9. Let $\succeq$ be represented by ψ and u as defined in Proposition 1, then for any act $g \in \mathcal{F}$ and all $f \in \mathcal{F}_{Co}(g)$ it holds that $U(f) = \nabla\psi(G)F$, where $F = u \circ f$ and $G = u \circ g$.*

Proof. Fix some $g \in \mathcal{F}$ and take any $f \in \mathcal{F}_{Co}(g)$. By definition of $\mathcal{F}_{Co}(g)$ the two acts f and g are comonotonic. Again, we show that $\nabla\psi(G)F = U(f)$:

$$\begin{aligned} \nabla\psi(G)F &= \psi^{'}(G, F) \\ &= \lim_{\alpha \downarrow 0} \frac{\psi(G + \alpha F) - \psi(G)}{\alpha} \\ &= \lim_{\alpha \downarrow 0} \frac{\psi(G) + \alpha\psi(F) - \psi(G)}{\alpha} \\ &= \psi(F) \\ &= U(f), \end{aligned}$$

where the third equality follows by Comonotonic Additivity. □

Since we have shown that Axioms 5 and 6 are weaker than Axiom 9, we can represent a decision makers' preferences also via Theorem 2. Note that there are $n!$ orderings of the n states in Ω. Figure 3 illustrates the resulting partition of utility vectors for the three state case. Thus, the set of acts $\mathcal{F}$ can be partitioned into $n!$ sets such that $\mathcal{F} = \cup_{i=1}^{n!} \mathcal{F}_{Co}(g_i)$, where each $g_i \in \mathcal{F}$ represents another strict ordering of states. By Proposition 8, each act in a set $\mathcal{F}_{Co}(g_i)$ can be evaluated by the same prior and thus the set of priors contains at most $n!$ different priors. Since both representations are equivalent, we can simply interpret the capacity v as a set of $n!$ priors.

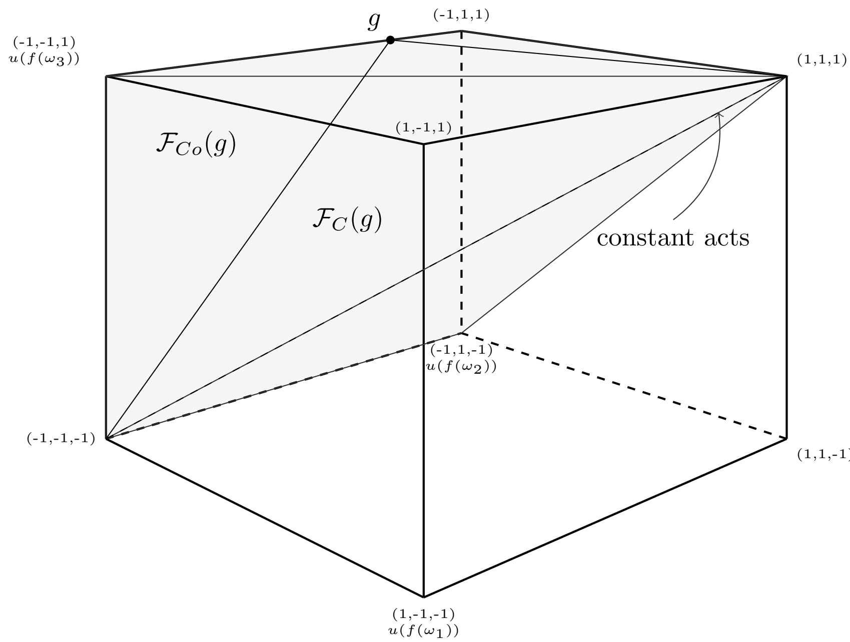

Figure 4: Example of C-independent region in a comonotonic region for three states.

Next we discuss the relationship between C-independent and comonotonic regions. First we show that for any $g \in \mathcal{F}$ we have $\mathcal{F}_C(g) \subseteq \mathcal{F}_{Co}(g)$, a fact that is also illustrated in Figure 4. Take $f \in \mathcal{F}_C(g)$ and note that f and g are comonotonic (as seen in the proof of Proposition 7). Thus, $f \in \mathcal{F}_{Co}(g)$ and therefore $\mathcal{F}_C(g) \subseteq \mathcal{F}_{Co}(g)$. The converse, however, is not true since not every act that is comonotonic with g is in $\mathcal{F}_C(g)$. In the two-state case, we have that $\mathcal{F}_C(g) = \mathcal{F}_{Co}(g)$ and since there are just two comonotonic regions we only require two priors. Moreover, if $n > 2$ we can take an act $g^{'} \in \mathcal{F}$ such that $g^{'}$ is comonotonic to g and such that $\mathcal{F}_C(g) \cap \mathcal{F}_C(g^{'}) = \emptyset$. Therefore $g^{'} \in \mathcal{F}_{Co}(g)$ but $g^{'} \notin \mathcal{F}_C(g)$. However, for any $f \in \mathcal{F}_C(g^{'})$ we have that $f \in \mathcal{F}_{Co}(g)$. Hence, we have found two sets $\mathcal{F}_C(g), \mathcal{F}_C(g^{'}) \subseteq \mathcal{F}_{Co}(g)$. This shows that there can be multiple C-independent regions in a comonotonic region.

5.3 Anscombe-Aumann

Anscombe and Aumann [1963] extended Von Neumann and Morgenstern's [1944] Risk Independence fully to acts. The result is the well-known Full Independence Axiom. Again, from here on we assume that $\succeq$ satisfies Axiom 10.

Axiom 10 (Full Independence)**.** *For any three acts $f, g, h \in \mathcal{F}$ and any $\alpha \in (0, 1)$, we have that $f \succeq g$ if and only if $\alpha f + (1-\alpha)h \succeq \alpha g + (1-\alpha)h$.*

Full Independence enables the standard subjective expected utility representation that is vastly used in economics. Its power is rooted in the fact that it induces Full Additivity of ψ and therefore also of the utility function U.

Proposition 9 (Full Additivity)**.** *Assume that* $\succeq$ *satisfies Axioms 1-4 and Axiom 10. Let* $\succeq$ *be represented by* ψ *and* u *as described in Proposition 1, then for any two acts* $f, g \in \mathcal{F}$ *and* $F, G \in [-1,1]^\Omega$ *such that* $u \circ f = F$ *and* $u \circ g = G$*, it holds that* $\psi(F + \alpha G) = \psi(F) + \alpha\psi(G)$ *for any* $\alpha > 0$ *with* $F + \alpha G \in [-1,1]^\Omega$*.*

Given Full Additivity we can easily show that a single prior is sufficient to obtain the utility of all acts.

Proposition 10. *Assume that* $\succeq$ *satisfies Axioms 1-4 and Axiom 10. Let* $\succeq$ *be represented by* ψ *and* u *as described in Proposition 1, then for any act* $g \in \mathcal{F}$ *and all* $f \in \mathcal{F}$ *it holds that* $U(f) = \nabla\psi(G)F$*, where* $F = u \circ f$ *and* $G = u \circ g$*.*

Proof. We have to show that $U(f) = \nabla\psi(G)F$ holds for any $f, g \in \mathcal{F}$ and $\nabla\psi(G) \in \mathcal{P}_G$. We assume $\succeq$ satisfies Axiom 10 and thereby that ψ is Fully Additive and homogenous. Then take any $f, g \in \mathcal{F}$ and note that

$$\begin{aligned} \nabla\psi(G)F &= \psi^{'}(G, F) \\ &= \lim_{\alpha \downarrow 0} \frac{\psi(G + \alpha F) - \psi(G)}{\alpha} \\ &= \lim_{\alpha \downarrow 0} \frac{\psi(G) + \alpha\psi(F) - \psi(G)}{\alpha} \\ &= \psi(F) \\ &= U(f), \end{aligned}$$

where the third equality holds by Full Additivity. □

Since Full Independence clearly implies Risk Independence as well as Independence of Certainty Equivalents, we can also represent the decision makers preferences using Theorem 2. Furthermore, if we combine Proposition 10 with Theorem 2 it is immediate that we only need one prior to represent the decision makers' preferences and thus we obtain the classical Anscombe-Aumann representation.

6 Discussion

Our approach is not the first to characterize the set of priors under weaker independence axioms. We relate our work to previous attempts and show how they connect. Next to the characterization of the set of priors, we pinpoint the exact independence assumption we need to obtain preferences that can be represented by MEU. The literature discusses even weaker assumptions on independence of acts and the implications. We show what happens in our setting if we relax C-Independence even further.

6.1 Characterizations of the Set of Priors

Given Axioms 1 to 6, we can characterize the set of priors by using the Gâteaux derivative of the function ψ. Ghirardato et al. [2004] use a different approach to characterize the set of priors. They define an "unambiguously preferred to" relationship, which is induced by $\succeq$.

Definition 5. Let $f, g \in \mathcal{F}$. Then f is unambiguously preferred to g, denote $f \succeq^{'} g$, if

$$\lambda f + (1-\lambda)h \succeq \lambda g + (1-\lambda)h$$

for all $\lambda \in (0,1]$ and all $h \in \mathcal{F}$.

It is clear that $\succeq^{'}$ is in general an incomplete relationship. They show that $\succeq^{'}$ can be represented in the style of Bewley [2002] using a set of priors.

Proposition 11. *Assume that $\succeq$ satisfies Axioms 1-4 and Axiom 7. Let $\succeq$ be represented by u as described in Proposition 1, then there exists a unique nonempty, weak compact*[5] *and convex set $\mathcal{C}$ of priors on Ω such that for all $f, g \in \mathcal{F}$,*

$$f \succeq^{'} g \iff \int_{\Omega} u(f)dp \geq \int_{\Omega} u(g)dp \text{ for all } p \in \mathcal{C}.$$

Furthermore, they show that the set of priors $\mathcal{C}$ is equal to the Clarke differential of ψ at 0, which is a generalized derivative for functions that are not differentiable. An important difference to a derivative is that the Clarke differential is set valued. Since $\psi(F) \in [-1,1]$ for all $F \in [-1,1]^{\Omega}$, it is locally Lipschitz and therefore by Ghirardato et al.'s [2004] Lemma A.7 we have that $\partial\psi(0) = \mathcal{P}$, where $\partial\psi$ is the Clarke differential of ψ and $\mathcal{P}$ is as defined in Section 5.1. Hence, both representations identify the same set of relevant priors.

6.2 Further Weakening of the C-Independence Axiom

Axiom 7, C-Independence, entails two features: (i) invariance to rescaling of utilities (by choosing $\langle p \rangle$ with $u(p) = 0$) and (ii) invariance to translations of utility profiles. Recall that a preference relation $\succeq$ satisfying Axioms 1 to 5 allows a representation as in Proposition 1. Nevertheless, by requiring (i) we can only show that ψ is positively homogeneous, which is insufficient to show that $\nabla\psi(F)$ is a prior for all $F \in [-1,1]^{\Omega}$. The same problem prevails when we completely dispose of any independence assumption on acts. Thus C-Additivity, which is ensured by (ii) as we will show below, seems to be necessary to obtain a simple prior representation. Maccheroni et al. [2006] present the following axiom that guarantees (ii) but not necessarily (i).

Axiom 11 (Weak Certainty Independence)**.** *If $f, g \in \mathcal{F}$, $p, q \in \Delta(X)$, and $\lambda \in (0,1)$ then $\lambda f + (1-\lambda)\langle p \rangle \succeq \lambda g + (1-\lambda)\langle p \rangle$ implies $\lambda f + (1-\lambda)\langle q \rangle \succeq \lambda g + (1-\lambda)\langle q \rangle$.*

Thus, given that an act mixed with a constant act is preferred to another act mixed with the same constant act, the first act is always preferred to the second when mixed in the same proportion with another constant act. For a preference relation $\succeq$ that satisfies Axioms 1 to 5 Proposition 1 holds and Axiom 11 induces C-Additivity of the function ψ but not homogeneity.

Proposition 12 (C-Additive)**.** *Assume that $\succeq$ satisfies Axioms 1-5 and Axiom 11. Let $\succeq$ be represented by ψ and u as described in Proposition 1, then $\psi(F + \alpha\mathbb{1}_{\Omega}) = \psi(F) + \alpha\psi(\mathbb{1}_{\Omega})$ for $\alpha > 0$ where $F + \alpha\mathbb{1}_{\Omega} \in [-1,1]^{\Omega}$.*

The lack of homogeneity of ψ prevents an equivalent of Lemma 2, which establishes that the $\psi(F)$ can be computed by $\psi^{'}(F,F)$ for $F \in [-1,1]^{\Omega}$. However, we can use the following generalized Mean Value Theorem to show that $\psi(F) = \psi^{'}(F - sF, F)$ for some $s \in (0,1)$ by simply letting $G := -F$ for $F \in \text{int}\left([-1,1]^{\Omega}\right)$. For some F that is not in the interior of $[-1,1]^{\Omega}$, we can approximate the utility by taking a sequence in int $\left([-1,1]^{\Omega}\right)$ that converges to F, which is possible by Continuity.

[5] Compact with respect to the weak topology.

Theorem 6 (Generalized Mean Value Theorem)**.** *Let K be a nonempty open subset of $[-1,1]^\Omega$ and ψ be Gâteaux differentiable with Gâteaux derivative $\psi^{'}(F,G)$ at $F \in [-1,1]^\Omega$ in the direction of $G \in [-1,1]^\Omega$. Then for any $F \in K$ and $F+G \in K$, there exists some $s \in (0,1)$ such that*

$$\psi(F+G) - \psi(F) = \psi^{'}(F+sG, G).$$

Proof. For the proof consult Ansari et al. [2013]. □

By Lemma 1 we know that $\psi^{'}(F-sF,F)$ can be represented as $\nabla\psi(F-sF)F$, if the Gâteaux derivative of ψ exists at $F-sF$. Again, it turns out that $\nabla\psi(F-sF)$ is a prior.

Proposition 13. *Assume that $\succeq$ satisfies Axioms 1-5 and Axiom 11. Let $\succeq$ be represented by ψ and u as described in Proposition 1, then if the Gâteaux derivative of ψ at $F \in [-1,1]^\Omega$ exists, the gradient $\nabla\psi(F)$ of ψ has the following properties:*

- $\nabla\psi(F)(\omega) \geq 0$ *for all* $\omega \in \Omega$, *and*
- $\sum_{\omega\in\Omega} \nabla\psi(F)(\omega) = 1$.

The proof of Proposition 13 is equivalent to that of Proposition 3 since the proof only uses Monotonicity, homogeneity of $\psi^{'}$ in the second component and C-Additivity. The following representation theorem is thus immediate.

Theorem 7. *Assume that $\succeq$ satisfies Axioms 1-5 and Axiom 11. Let $\succeq$ be represented by ψ and u as described in Proposition 1, then if $F \in int\left([-1,1]^\Omega\right)$, the Gâteaux derivative of ψ at F exists, and $F = u \circ f$ for some $f \in \mathcal{F}$, then*

$$U(f) = \nabla\psi(\hat{F})F,$$

where $\nabla\psi(\hat{F})$ is the gradient at $\hat{F} := F - sF$ for some $s \in (0,1)$. Moreover, $\nabla\psi(\hat{F})$ is a probability measure.

Hence, the decisions maker still evaluates acts using a prior, however, we cannot say which prior exactly since s is possibly different for every act $f \in \mathcal{F}$.

Appendix

A1. Proofs Section 2

From now on let $\succeq$ be the agent's preference relation over $\mathcal{F}$. Then let $\succeq^*$ be the preference relation over $\Delta(X)$ induced by $\succeq$ such that for any $p, q \in \Delta(X)$ we have $p \succeq^* q$ if and only if $\langle p \rangle \succeq \langle q \rangle$.

Proposition 14. *Assume that $\succeq$ satisfies Axioms 1-5, then the preference relation $\succeq^*$ induced by $\succeq$ satisfies Weak Order, Continuity, and Risk Independence with respect to lotteries in $\Delta(X)$.*

Proof. The properties of $\succeq^*$ follow directly from the definition. □

Proposition 15. *Assume that $\succeq$ satisfies Axioms 1-5, then there exist $p_*, p^* \in \Delta(X)$ such that for all acts $f \in \mathcal{F}$ we have that $\langle p^* \rangle \succeq f \succeq \langle p_* \rangle$.*

Proof. Let $x^*, x_* \in X$ be such that $\langle x^* \rangle \succeq \langle x \rangle \succeq \langle x_* \rangle$ for all $x \in X$, where $\langle x \rangle$ is the constant act with $\langle x \rangle(\omega)(x) = 1$ for all $\omega \in \Omega$. Then by Axiom 5 it follows that $\langle x^* \rangle \succeq \langle p \rangle \succeq \langle x_* \rangle$ for all $p \in \Delta(X)$. By Axiom 2, $\langle x^* \rangle \succeq f \succeq \langle x_* \rangle$ for all $f \in \mathcal{F}$. Set $\langle p^* \rangle := \langle x^* \rangle$ and $\langle p_* \rangle := \langle x_* \rangle$. Thus, we have for all acts $f \in \mathcal{F}$ that $\langle p^* \rangle \succeq f \succeq \langle p_* \rangle$. □

Proposition 16. *Assume that $\succeq$ satisfies Axioms 1-5, then there exists an affine map u on $\Delta(X)$ that represents $\succeq^*$.*

Proof. Since $\succeq^*$ satisfies *Weak Order, Continuity*, and *Risk Independence* it follows from the Von Neumann-Morgenstern Expected Utility Theorem [Von Neumann and Morgenstern, 1944] that there exists an affine map u on $\Delta(X)$ that represents $\succeq^*$. □

Proposition 17. *Assume that $\succeq$ satisfies Axioms 1-5, then for any $f \in \mathcal{F}$, there is a unique $\alpha_f \in [0, 1]$ such that $f \sim \alpha_f \langle p^* \rangle + (1 - \alpha_f)\langle p_* \rangle$.*

Proof. First we have to show that for any act $f \in \mathcal{F}$ there exists a mixture of constant acts $p^*, p_* \in \Delta(X)$ such that $f \sim \alpha\langle p^* \rangle + (1 - \alpha)\langle p_* \rangle$ for some $\alpha \in [0, 1]$. If $f \sim \langle p^* \rangle$ we simply take $\alpha = 1$ and if $f \sim \langle p_* \rangle$ we take $\alpha = 0$. In the case in which $\langle p^* \rangle \succ f \succ \langle p_* \rangle$, Axiom 3 guarantees that there exists a $\alpha \in (0, 1)$ such that $f \sim \alpha\langle p^* \rangle + (1 - \alpha)\langle p_* \rangle$.

Secondly, we have to show that the α for which $f \sim \alpha\langle p^* \rangle + (1 - \alpha)\langle p_* \rangle$ holds is unique. Take the acts $p^*, p_* \in \Delta(X)$. Then, $\langle p^* \rangle \succ \langle p_* \rangle$ due to Axiom 4. Subsequently, apply Axiom 5 to obtain for any $\lambda \in (0, 1)$

$$\lambda\langle p^* \rangle + (1 - \lambda)\langle p^* \rangle \succ \lambda\langle p_* \rangle + (1 - \lambda)\langle p^* \rangle,$$

where the left-hand side can simply be expressed as $\langle p^* \rangle$. Then we apply Axiom 5 again and obtain for any $\gamma \in (0, 1)$

$$\gamma\langle p^* \rangle + (1 - \gamma)\langle p_* \rangle \quad \succ \quad (\gamma(1 - \lambda))\,\langle p^* \rangle + (1 - \gamma(1 - \lambda))\,\langle p_* \rangle. \tag{3}$$

Now we take any $\alpha, \beta \in (0, 1)$ and assume w.l.o.g. $\alpha > \beta$. Then take $\gamma = \alpha$ and $\lambda = (\gamma - \beta)/\gamma$. Then $\beta = \gamma(1 - \lambda)$ and equation (3) implies that $\alpha\langle p^* \rangle + (1 - \alpha)\langle p_* \rangle \succ \beta\langle p^* \rangle + (1 - \beta)\langle p_* \rangle$. Since this is true for any $\alpha > \beta$ we can conclude that α is unique. □

Proposition 18. *Assume that $\succeq$ satisfies Axioms 1-5, then there exists a unique $U : \mathcal{F} \to \mathbb{R}$ that represents $\succeq$ and satisfies*

$$U\left(\langle p \rangle\right) = u\left(p\right) \tag{4}$$

for all $p \in \Delta(X)$.

Proof. Define U on $\mathcal{F}$ by $U(f) := u(\alpha_f p^* + (1-\alpha_f)p_*)$, where α_f is found as in Proposition 17. It's clear that U represents $\succeq$ and satisfies (4). Moreover, if V was another such function, we would have for any $f \in \mathcal{F}$,

$$U(f) = U(\langle p_f \rangle) = u(p_f) = V(\langle p_f \rangle) = V(f),$$

where $p_f = \alpha_f p^* + (1-\alpha_f)p_*$. □

From now on we assume $u(p^*) = 1$ and $u(p_*) = -1$.

Proposition 19. *Assume that $\succeq$ satisfies Axioms 1-5, then* $\{u \circ f : f \in \mathcal{F}\} = [-1,1]^\Omega$.

Proof. For the "$\subseteq$"-direction remember that $u : \Delta(X) \to [-1,1]$. Since the composition is a state-wise operation, we have $u \circ f \in [-1,1]^\Omega$ for all $f \in \mathcal{F}$.

Now consider the "$\supseteq$"-direction. Take a $F \in [-1,1]^\Omega$. Then since Axiom 3 holds for u by the Mean Value Theorem there exists a $p_\omega \in \Delta(X)$ such that $u(p_\omega) = F(\omega)$ for every $\omega \in \Omega$. Let $g \in \mathcal{F}$ be such that $g(\omega) := p_\omega$ and therefore we have $u \circ g = F$. □

Definition 6. Let the function $\psi : [-1,1]^\Omega \to \mathbb{R}$ be such that $\psi(u \circ f) := U(f)$ for all $f \in \mathcal{F}$.

Such a function ψ must exists because Axiom 2 implies that $U(f) = U(g)$ whenever $u \circ f = u \circ g$. Since U represents $\succeq$ and we defined $\psi(u \circ f) = U(f)$ for all $f \in \mathcal{F}$, the function ψ is clearly monotonic and continuous.

Proof Proposition 1. By Proposition 16 there exists an affine map u on $\Delta(X)$ that represents $\succeq^*$. Take $p^*, p_* \in \Delta(X)$ as in Proposition 15 and define $u(p^*) = 1$ and $u(p_*) = -1$. Then according to Proposition 18 there exists a utility function U on $\mathcal{F}$ that represents $\succeq$ and is such that $U(\langle p \rangle) = u(p)$ for all $p \in \Delta(X)$. Moreover, we know that there exists a function $\psi : [-1,1]^\Omega \to \mathbb{R}$ such that $\psi(u \circ f) := U(f)$ for all $f \in \mathcal{F}$. Since ψ also represents $\succeq$ it is continuous and monotonic. Finally, it follows directly from Proposition 17 that for all $f \in \mathcal{F}$ there is a $p \in \Delta(X)$ such that $f \sim \langle p \rangle$. □

A2. Proofs Section 3

Proof Lemma 1. Take $F \in [-1,1]^\Omega$ where the Gâteaux derivative of ψ exists and any $G \in [-1,1]^\Omega$. Then,

$$\begin{aligned} \nabla\psi(F)G &= \sum_{\omega \in \Omega} \psi^{'}(F, \mathbb{1}_\omega)G(\omega) \\ &= \psi^{'}(F, \sum_{\omega \in \Omega} G(\omega)\mathbb{1}_\omega) \\ &= \psi^{'}(F, G), \end{aligned}$$

where the second equality holds by homogeneity of $\psi^{'}$ in the second component. □

A3. Proofs Section 4

Proof Proposition 2. For the first property take any $f \in \mathcal{F}$ and $F \in [-1,1]^\Omega$ such that $u \circ f = F$. Let $g := \lambda f + (1-\lambda)\langle p\rangle$ with $u(p) = 0$ and $\lambda \in (0,1]$. Note that

$$\begin{aligned} u \circ g &= u \circ (\lambda f + (1-\lambda)\langle p\rangle) \\ &= \lambda (u \circ f) + (1-\lambda) u \circ \langle p\rangle \\ &= \lambda (u \circ f), \end{aligned}$$

where the second equality holds by affinity of u. Applying ψ yields $\psi(u \circ g) = \psi(\lambda(u \circ f)) = \psi(\lambda F)$. By Proposition 1 we can find a $q \in \Delta(X)$ such that $f \sim \langle q\rangle$. Thus, Axiom 6 implies $g \sim \lambda\langle q\rangle + (1-\lambda)\langle p\rangle$. The utility of g can then be expressed as

$$\begin{aligned} U(g) &= u(\lambda q + (1-\lambda)p) \\ &= \lambda u(q) \\ &= \lambda U(\langle q\rangle) \\ &= \lambda U(f), \end{aligned}$$

where again the second equality follows by affinity of u. Now we can show that $U(g) = \lambda\psi(F)$ by noting that

$$U(g) = \lambda U(f) = \lambda\psi(u \circ f) = \lambda\psi(F),$$

and thus that

$$\psi(\lambda F) = \psi(\lambda(u \circ f)) = \psi(u \circ g) = U(g) = \lambda\psi(F),$$

for all $\lambda \in (0,1]$. Finally, we show that $\psi(\lambda F) = \lambda\psi(F)$ holds for all $\lambda > 1$ as well. Take any $F \in [-1,1]^\Omega$ and $\lambda > 1$ such that $\lambda F \in [-1,1]^\Omega$. Then it follows from the above that $\psi(F) = \psi(\frac{1}{\lambda}\lambda F) = \frac{1}{\lambda}\psi(\lambda F)$ and hence $\lambda\psi(F) = \psi(\lambda F)$ for all $\lambda > 1$.

To prove the second property take any $f \in \mathcal{F}$ and $F \in [-1,1]^\Omega$ such that $u \circ f = F$. Furthermore, take $\alpha \in [0,1]$ and $p \in \Delta(X)$ such that $u(p) = \alpha$. By Proposition 1 we can find $q \in \Delta(X)$ such that $f \sim \langle q\rangle$. Therefore, Axiom 6 implies $\frac{1}{2}f + \frac{1}{2}\langle p\rangle \sim \frac{1}{2}\langle q\rangle + \frac{1}{2}\langle p\rangle$. It follows immediately that $U(\frac{1}{2}f + \frac{1}{2}\langle p\rangle) = U(\frac{1}{2}\langle q\rangle + \frac{1}{2}\langle p\rangle)$. We now use positive homogeneity of ψ and affinity of u to show

$$\begin{aligned} 2U\left(\frac{1}{2}f + \frac{1}{2}\langle p\rangle\right) &= 2\psi\left(u \circ \left(\frac{1}{2}f + \frac{1}{2}\langle p\rangle\right)\right) \\ &= \psi(u \circ f + u \circ \langle p\rangle) \\ &= \psi(F + \alpha\mathbb{1}_\Omega). \end{aligned}$$

Furthermore, we can use u's affinity and the definitions of U and ψ to show

$$\begin{aligned} 2U\left(\frac{1}{2}\langle q\rangle + \frac{1}{2}\langle p\rangle\right) &= 2u\left(\frac{1}{2}q + \frac{1}{2}p\right) \\ &= u(q) + \alpha \\ &= U(\langle q\rangle) + \alpha \\ &= U(f) + \alpha \\ &= \psi(F) + \alpha. \end{aligned}$$

Combining our two findings and using the fact that $U(\frac{1}{2}f + \frac{1}{2}\langle p\rangle) = U(\frac{1}{2}\langle q\rangle + \frac{1}{2}\langle p\rangle)$ we obtain $\psi(F + \alpha\mathbb{1}_\Omega) = \psi(F) + \alpha\psi(\mathbb{1}_\Omega)$ since $\psi(\mathbb{1}_\Omega) = 1$. Finally, we show that $\psi(F + \alpha\mathbb{1}_\Omega) = \psi(F) + \alpha$ holds for all $\alpha > 1$ as well. Take $F \in [-1, 1]^\Omega$ and $\alpha > 1$ such that $F + \alpha\mathbb{1}_\Omega \in [-1, 1]^\Omega$. Note that $\alpha \leq 2$. Let $G := F + \mathbb{1}_\Omega$, then we have

$$\begin{aligned} \psi(F + \alpha\mathbb{1}_\Omega) &= \psi(F + \mathbb{1}_\Omega - \mathbb{1}_\Omega + \alpha\mathbb{1}_\Omega) \\ &= \psi(G + (\alpha - 1)\mathbb{1}_\Omega) \\ &= \psi(G) + (\alpha - 1) \\ &= \psi(F + \mathbb{1}_\Omega) + (\alpha - 1) \\ &= \psi(F) + \alpha, \end{aligned}$$

where the third and fifth equality hold by the result obtained above. □

Proof Lemma 2. We start by considering the gradient of ψ at any $F \in [-1, 1]^\Omega$ where the Gâteaux derivative exists. We can use the definition $\psi^{'}$ and the homogeneity of ψ to show that

$$\begin{aligned} \psi^{'}(F, F) &= \lim_{\alpha\downarrow 0} \frac{\psi(F + \alpha F) - \psi(F)}{\alpha} \\ &= \lim_{\alpha\downarrow 0} \frac{\psi((1 + \alpha)F) - \psi(F)}{\alpha} \\ &= \lim_{\alpha\downarrow 0} \frac{(1 + \alpha)\psi(F) - \psi(F)}{\alpha} \\ &= \psi(F). \end{aligned}$$

□

Proof Proposition 3. Consider the first property and therefore the gradient of ψ at $F \in [-1, 1]^\Omega$ at state $\omega \in \Omega$:

$$\begin{aligned} \nabla\psi(F)(\omega) &= \psi^{'}(F, \mathbb{1}_\omega) \\ &= \lim_{\alpha\downarrow 0} \frac{\psi(F + \alpha\mathbb{1}_\omega) - \psi(F)}{\alpha}. \end{aligned}$$

By Axiom 2 we have $\psi(F + \alpha\mathbb{1}_\omega) \geq \psi(F)$ for all $\alpha \geq 0$ and thus the limit needs to be larger or equal to zero. It follows that $\nabla\psi(F)(\omega) \geq 0$.

For the second property, consider the sum

$$\begin{aligned}
\sum_{\omega\in\Omega} \nabla\psi(F)(\omega) &= \sum_{\omega\in\Omega} \psi^{'}(F, \mathbb{1}_\omega) \\
&= \psi^{'}(F, \sum_{\omega\in\Omega} \mathbb{1}_\omega) \\
&= \psi^{'}(F, \mathbb{1}_\Omega) \\
&= \lim_{\alpha\downarrow 0} \frac{\psi(F+\alpha\mathbb{1}_\Omega)-\psi(F)}{\alpha} \\
&= \lim_{\alpha\downarrow 0} \frac{\psi(F)+\alpha\psi(\mathbb{1}_\Omega)-\psi(F)}{\alpha} \\
&= \lim_{\alpha\downarrow 0} \frac{\alpha\psi(\mathbb{1}_\Omega)}{\alpha} \\
&= \psi(\mathbb{1}_\Omega) = 1.
\end{aligned}$$

The second equality follows by homogeneity of $\psi^{'}$ in the second component and the fifth by C-Additivity of ψ. □

Proof Proposition 4. Take any act $f \in \mathcal{F}$ and $F \in [-1,1]^\Omega$ such that $u \circ f = F$, where the derivative of ψ at F does not exist. Since ψ is continuous and the derivative of ψ exists almost everywhere, we can approximate the utility of f by taking a sequence $(F_n)_{n\in\mathbb{N}}$ such that the Gâteaux derivative exists at all F_n for all n and $F_n \in [-1,1]^\Omega$. Furthermore, we require $\lim_{n\to\infty} F_n = F$. Then, by continuity of ψ, $\psi(F) = \lim_{n\to\infty} \psi(F_n) = \lim_{n\to\infty} \nabla\psi(F_n)F_n$, where the last equality follows from Theorem 2. □

Proof of Theorem 3. The "only if" direction follows from Theorem 2, and Proposition 1. Thus we only have to prove the "if" direction. Axiom 1 (Weak Order) is implied by the existence of a utility function U.

For Axiom 2 (Monotonicity) take $f, g \in \mathcal{F}$ such that $\langle f(\omega)\rangle \succeq \langle g(\omega)\rangle$ for all $\omega \in \Omega$ and let $F = u \circ f$ and $G = u \circ g$. Then $F(\omega) \geq G(\omega)$ for all $\omega \in \Omega$. By monotonicity of ψ we have $\psi(F) \geq \psi(G)$ and since $\psi(F) = U(f)$ for all $f \in \mathcal{F}$ it follows that $f \succeq g$.

To show that $\succeq$ satisfies Axiom 3, take $f, g, h \in \mathcal{F}$ such that $f \succ g \succ h$. We have to show that there is an $\alpha \in (0,1]$ such that $U(g) = U(\alpha f + (1-\alpha)g)$. Let $F = u \circ f$, $G = u \circ g$, $H = u \circ h$. Then, we must show that there is an $\alpha \in (0,1]$ such that $\psi(G) = \psi(\alpha F + (1-\alpha)H)$. This, however, follows from continuity of ψ and the fact that $\psi(F) > \psi(G)$ as well as $\psi(H) < \psi(G)$.

Axiom 4 simply follows from the assumption that there exist $p_*, p^* \in \Delta(X)$ such that $u(p^*) = 1$ and $u(p_*) = -1$.

For Axiom 5 take $x, y, z \in \Delta(X)$ such that $\langle x\rangle \succ \langle y\rangle$. Moreover, let $u(x) = \alpha$, $u(y) = \beta$, and $u(z) = \gamma$. Take some $\lambda \in (0,1]$ and note that

$$\begin{aligned}
U(\lambda\langle x\rangle + (1-\lambda)\langle z\rangle) &= \psi(u \circ (\lambda\langle x\rangle + (1-\lambda)\langle z\rangle) \\
&= \psi(\lambda u \circ \langle x\rangle + (1-\lambda) u \circ \langle z\rangle) \\
&= \psi(\lambda\alpha\mathbb{1}_\Omega + (1-\lambda)\gamma\mathbb{1}_\Omega) \\
&= \lambda\psi(\alpha\mathbb{1}_\Omega) + (1-\lambda)\psi(\gamma\mathbb{1}_\Omega)) \\
&= \lambda U(\langle x\rangle) + (1-\lambda)U(\langle z\rangle) \\
&> \lambda U(\langle y\rangle) + (1-\lambda)U(\langle z\rangle) \\
&= \lambda\psi(\beta\mathbb{1}_\Omega) + (1-\lambda)\psi(\gamma\mathbb{1}_\Omega)) \\
&= \psi(\lambda\beta\mathbb{1}_\Omega + (1-\lambda)\gamma\mathbb{1}_\Omega) \\
&= U(\lambda\langle y\rangle + (1-\lambda)\langle z\rangle),
\end{aligned}$$

where the second equality holds by affinity of u and the fourth and seventh equality hold by positive homogeneity and C-Additivity of ψ, implying $\lambda\langle x\rangle + (1-\lambda)\langle z\rangle \succ \lambda\langle y\rangle + (1-\lambda)\langle z\rangle$.

Finally, for Axiom 6 we have to show that if $f \sim \langle p\rangle$, where $f \in \mathcal{F}$ and $p \in \Delta(X)$, then $U(\lambda f + (1-\lambda)\langle q\rangle) = U(\lambda\langle p\rangle + (1-\lambda)\langle q\rangle)$ for all $q \in \Delta(X)$ and $\lambda \in [0,1)$. Take $f \in \mathcal{F}$ and $p \in \Delta(X)$ with $f \sim \langle p\rangle$ and $u(p) = \alpha$, and some $q \in \Delta(X)$ such that $u(q) = \beta$. Let $F = u \circ f$, $\lambda \in [0,1)$ and note that

$$\begin{aligned}
U(\lambda f + (1-\lambda)\langle q\rangle) &= \psi(\lambda F + (1-\lambda)\beta\mathbb{1}_\Omega) \\
&= \lambda\psi(F) + \psi((1-\lambda)\beta\mathbb{1}_\Omega) \\
&= \lambda U(f) + U((1-\lambda)\langle q\rangle) \\
&= \lambda U(\langle p\rangle) + U((1-\lambda)\langle q\rangle) \\
&= \lambda\psi(\alpha\mathbb{1}_\Omega) + \psi((1-\lambda)\beta\mathbb{1}_\Omega) \\
&= \psi(\lambda\alpha\mathbb{1}_\Omega + (1-\lambda)\beta\mathbb{1}_\Omega) \\
&= U(\lambda\langle p\rangle + (1-\lambda)\langle q\rangle),
\end{aligned}$$

where the second and the sixth equality hold by positive homogeneity and C-Additivity of ψ. □

A4. Proofs Section 5

Proof of Proposition 5. First of all, take any $f, g \in \mathcal{F}$ such that $f \sim g$. Then by Axiom 8 we have that $\frac{1}{2}f + \frac{1}{2}g \succeq f$. Let $F := u \circ f$ and $G := u \circ g$, then it follows that

$$\begin{aligned}
\psi(\frac{1}{2}F + \frac{1}{2}G) &= \psi(u \circ (\frac{1}{2}f + \frac{1}{2}g)) \\
&= U(\frac{1}{2}f + \frac{1}{2}g) \\
&\geq U(f) \\
&= \psi(F) \\
&= \frac{1}{2}\psi(F) + \frac{1}{2}\psi(G).
\end{aligned}$$

Then by positive homogeneity of ψ we have that $\psi(F+G) \geq \psi(F) + \psi(G)$. Now consider $f, g \in \mathcal{F}$ such that $f \succ g$. Let $\alpha := \psi(F) - \psi(G)$ and define $H := G + \alpha\mathbb{1}_\Omega$. By C-Additivity and the fact that $\psi(\mathbb{1}_\Omega) = 1$ we have $\psi(H) = \psi(G) + \alpha = \psi(F)$. Thus by our previous result it holds that

$$\psi(F + G + \alpha\mathbb{1}_\Omega) = \psi(F + H) \geq \psi(F) + \psi(H) = \psi(F) + \psi(G) + \alpha.$$

Then by C-Additivity and homogeneity of ψ we have $\psi(F + G + \alpha\mathbb{1}_\Omega) = \psi(F + G) + \psi(\alpha\mathbb{1}_\Omega) = \psi(F + G) + \alpha$. Therefore, it holds that $\psi(F + G) \geq \psi(F) + \psi(G)$. □

Proposition 20. *Let $\succeq$ be a preference relation on $\mathcal{F}$ satisfying Axioms 1 to 5, and Axiom 9 then it holds that for all pairwise comonotonic acts $f, g, h \in \mathcal{F}$ with $f \succeq g$ that $\alpha f + (1-\alpha)h \succeq \alpha g + (1-\alpha)h$ for all $\alpha \in (0, 1)$.*

Proof. First of all, note that Axiom 9 implies Axioms 5 and 6 and thus Theorem 2 holds. Then, if there is no $g \in \mathcal{F}$ with $g \succ f$, take any $f, g, h \in \mathcal{F}$ such that f, g, h are pairwise comonotonic and $f \sim g$. Note that both f and g must be constant acts and therefore $u \circ f = u \circ g$. This holds because f must be the act that gives 1 in every state, otherwise there would be an $h \in \mathcal{F}$ such that $h \succ f$. Then either g is a constant act or we take g's certainty equivalent and simply call it g. Since $f \sim g$ the constant acts must be equal. Hence, we must have that $u \circ (\lambda f + (1-\lambda)h) = u \circ (\lambda g + (1-\lambda)h)$ for $\lambda \in (0, 1)$. This implies $\psi\left(u \circ (\lambda f + (1-\lambda)h)\right) = \psi\left(u \circ (\lambda g + (1-\lambda)h)\right)$, which by affinity of u is equivalent to $\psi\left(\lambda u \circ f + (1-\lambda)u \circ h\right) = \psi\left(\lambda u \circ g + (1-\lambda)u \circ h\right)$. Hence, we have that $U(\lambda f + (1-\lambda)h) = U(\lambda g + (1-\lambda)h)$ and therefore $\lambda f + (1-\lambda)h \sim \lambda g + (1-\lambda)h$. In the case where there is no $g \in \mathcal{F}$ such that $f \succ g$ then we can use the same argument.

Now consider the case where $f \in \mathcal{F}$ is in the interior of the set of acts and take any $f, g, h \in \mathcal{F}$ such that f, g and h are pairwise comonotonic and $f \sim g$. Then take two sequences such that $f_n^+ \to f$ and $f_n^- \to f$ as $n \to \infty$, such that f_n^+ and f_n^- are comonotonic to f. Moreover, we require that $f_n^+ \succ f$ and $f \succ f_n^-$ for all $n \in \mathbb{N}$. Thus, it must hold that $f_n^+ \succ g$ and therefore by Comonotonic Independence that $\lambda f_n^+ + (1-\lambda)h \succ \lambda g + (1-\lambda)h$ for any $\lambda \in (0, 1)$. Furthermore, it must hold that $g \succ f_n^-$ and therefore by Comonotonic Independence that $\lambda g + (1-\lambda)h \succ \lambda f_n^- + (1-\lambda)h$ for any $\lambda \in (0, 1)$. Hence, as $n \to \infty$ it must hold by Continuity that $f \sim g$ implies $\lambda f + (1-\lambda)h \sim \lambda g + (1-\lambda)h$ for any $\lambda \in (0, 1)$. This gives us the symmetric part of Comonotonic Independence and since the asymmetric part is already given in Axiom 9 it concludes the proof. □

Lemma 3. *Assume that $\succeq$ satisfies Axioms 1-6. Let $\succeq$ be represented by ψ and u as described in Proposition 1, then take any $f, g \in \mathcal{F}$ and $p \in \Delta(X)$ such that $u(p) = 0$. Let $h := \alpha g + (1-\alpha)\langle p \rangle$ for some $\alpha \in (0, 1)$ and $F, G, H \in [-1, 1]^\Omega$ such that $u \circ f = F$, $u \circ g = G$, and $u \circ h = H$. If it holds that $\frac{1}{2}f + \frac{1}{2}h \sim \frac{1}{2}f + \frac{1}{2}\langle q \rangle$, where $h \sim \langle q \rangle$ for some $q \in \Delta(X)$, then we have that*

$$\psi(F + \alpha G) = \psi(F) + \alpha\psi(G).$$

Proof. Note that $2U(\frac{1}{2}f + \frac{1}{2}h) = 2U(\frac{1}{2}f + \frac{1}{2}\langle q \rangle)$. First we take on the left-hand side and show that $2U(\frac{1}{2}f + \frac{1}{2}h) = \psi(F + \alpha G)$ by using the affinity of u and positive homogeneity of ψ:

$$\begin{aligned}
2U(\frac{1}{2}f + \frac{1}{2}h) &= 2\psi\left(u \circ (\frac{1}{2}f + \frac{1}{2}h)\right) \\
&= \psi(u \circ f + u \circ h) \\
&= \psi\left(u \circ f + (u \circ (\alpha g + (1-\alpha)\langle p \rangle))\right) \\
&= \psi\left(u \circ f + \alpha u \circ g\right) \\
&= \psi\left(F + \alpha G\right).
\end{aligned}$$

For the right-hand side we perform a similar exercise:

$$
\begin{aligned}
2U(\frac{1}{2}f + \frac{1}{2}\langle q\rangle) &= 2\psi(\frac{1}{2}F + \frac{1}{2}u \circ \langle q\rangle) \\
&= \psi(F + u \circ \langle q\rangle) \\
&= \psi(F) + \psi(u \circ \langle q\rangle) \\
&= \psi(F) + \psi\left(u \circ (\alpha g + (1-\alpha)\langle p\rangle)\right) \\
&= \psi(F) + \psi(\alpha u \circ g + (1-\alpha) u \circ \langle p\rangle) \\
&= \psi(F) + \alpha\psi(u \circ g) \\
&= \psi(F) + \alpha\psi(G).
\end{aligned}
$$

To obtain the second equality we use positive homogeneity of ψ, for the third we use C-Additivity of ψ, for the fifth we use affinity of u, and for the sixth equality we use positive homogeneity of ψ. Thus we showed that $\psi(F + \alpha G) = \psi(F) + \alpha\psi(G)$. □

Proof of Proposition 7. Define $h := \alpha g + (1-\alpha)\langle p\rangle$ for any $\alpha \in (0,1)$ where $p \in \Delta(X)$ is such that $u(p) = 0$. First we show that f and h are comonotonic. Take $H \in [-1,1]^\Omega$ such that $u \circ h = H$. Then we have to show that $\left(F(\omega) - F(\omega^{'})\right)\left(H(\omega) - H(\omega^{'})\right) \geq 0$ for $\omega, \omega^{'} \in \Omega$, a condition which is equivalent to the definition of Comonotonicity. Note that for any two states $\omega, \omega^{'} \in \Omega$ we have that

$$
\begin{aligned}
H(\omega) - H(\omega^{'}) &= (\alpha G(\omega) + (1-\alpha)u(p)) - (\alpha G(\omega^{'}) + (1-\alpha)u(p)) \\
&= \alpha(G(\omega) - G(\omega^{'})).
\end{aligned}
$$

Since $\alpha > 0$ it does not change the sign of $G(\omega) - G(\omega^{'})$ and thus h is comonotonic to g and therefore f to h. Furthermore, take $q \in \Delta(X)$ such that $\langle q\rangle \sim h$. By Proposition 20 we obtain $\frac{1}{2}f + \frac{1}{2}h \sim \frac{1}{2}f + \frac{1}{2}\langle q\rangle$. Now we can use Lemma 3 to show that $\psi(F + \alpha G) = \psi(F) + \alpha\psi(G)$ for any $\alpha \in (0,1)$. Then for $\alpha \geq 1$ and any $F, \alpha G \in [-1,1]^\Omega$ by positive homogeneity of ψ and the result above we have that

$$
\begin{aligned}
\psi(F + \alpha G) &= \psi(\alpha(\frac{1}{\alpha}F + G)) \\
&= \alpha\psi(\frac{1}{\alpha}F + G) \\
&= \alpha\psi(\frac{1}{\alpha}F) + \alpha\psi(G) \\
&= \psi(F) + \alpha\psi(G).
\end{aligned}
$$

Thus $\psi(F + \alpha G) = \psi(F) + \alpha\psi(G)$ holds for any $\alpha > 0$. □

Proof of Theorem 5. Define the capacity v by $v(X) := \psi(\mathbb{1}_X)$ for any $X \subseteq \Omega$, and $v(\emptyset) = 0$. Remember that $\mathbb{1}_X$ is an $1 \times n$ vector with 1's where the respective state is included in X. Our task is to show that v is the unique capacity such that (2) holds. For that matter, take any act $f \in \mathcal{F}$ and a $F \in [-1,1]^\Omega$ such that $u \circ f = F$. Recall that σ is a bijective function that orders the states of F from best to worst and define $O(i) := \{\sigma(1), \ldots, \sigma(i)\}$ as the set of ordered states of F up to rank i.

First of all, note that (1) can be rewritten as

$$\int_\Omega F dv = \sum_{i=1}^{n} F(\sigma(i)) \left(v\left(O(i)\right) - v\left(O(i-1)\right)\right). \tag{5}$$

Then consider

$$\begin{aligned} \psi(F) &= \nabla\psi(F)F \\ &= \sum_{\omega\in\Omega} \psi^{'}(F, \mathbb{1}_\omega)F(\omega) \\ &= \sum_{i=1}^{n} \psi^{'}(F, \mathbb{1}_{\sigma(i)})F(\sigma(i)), \end{aligned} \tag{6}$$

where the third inequality is just a reordering of the sum according to the payoff size of the states in F. Note that we have $\mathbb{1}_{\sigma(i)} = \mathbb{1}_{O(i)} - \mathbb{1}_{O(i-1)}$ and thus

$$\begin{aligned} \psi^{'}(F, \mathbb{1}_{\sigma(i)}) &= \psi^{'}(F, \mathbb{1}_{O(i)} - \mathbb{1}_{O(i-1)}) \\ &= \psi^{'}(F, \mathbb{1}_{O(i)}) - \psi^{'}(F, \mathbb{1}_{O(i-1)}) \end{aligned}$$

by homogeneity of $\psi^{'}$ in the second component. Let us now compute the Gâteaux derivative at F in the direction of $\mathbb{1}_{O(m)}$ for some $m \in \Omega$ by using the fact that F and $\mathbb{1}_{O(m)}$ are comonotonic:

$$\begin{aligned} \psi^{'}(F, \mathbb{1}_{O(m)}) &= \lim_{\alpha\downarrow 0} \frac{\psi(F + \alpha\mathbb{1}_{O(m)}) - \psi(F)}{\alpha} \\ &= \lim_{\alpha\downarrow 0} \frac{\psi(F) + \alpha\psi(\mathbb{1}_{O(m)}) - \psi(F)}{\alpha} \\ &= \psi(\mathbb{1}_{O(m)}), \end{aligned}$$

where the second equality follows from Proposition 7 (Comonotonic Additivity) and the fact that F and $\alpha\mathbb{1}_{O(m)}$ are comonotonic and thus

$$\psi^{'}(F, \mathbb{1}_{\sigma(i)}) = \psi(\mathbb{1}_{O(i)}) - \psi(\mathbb{1}_{O(i-1)}). \tag{7}$$

By combining (6), (7), and the definition of the capacity v we obtain (5). Hence, $U(f) = \psi(F) = \int_\Omega F dv$.

Next we show that v is indeed a capacity. We have $v(\emptyset) = 0$ by definition and $v(\Omega) = 1$ by the fact that $v(\Omega) = v(O(n)) = \psi(1_\Omega) = 1$. Moreover, we have to show that for any $X, Y \subseteq \Omega$ such that $Y \subset X$ we have $v(Y) \leq v(X)$. Since $v(Z) = \psi(\mathbb{1}_Z)$ for every $Z \subseteq \Omega$, it follows by monotonicity of ψ that $v(Y) \leq v(X)$.

Finally, we have to show that v is unique. Suppose there is another capacity w such that $U(f) = \int_\Omega F dw$ for all $f \in \mathcal{F}$ with $F = u \circ f$. Then take $F := \mathbb{1}_X$ for some $X \subseteq \Omega$ and let $k := |X|$.

Note that $X = \{\sigma(i), \ldots, \sigma(k)\}$. Then compute $U(f)$:

$$\begin{aligned}
U(f) &= \int_{\Omega} F dv \\
&= \sum_{i=1}^{n} F(\sigma(i))(v(O(i)) - v(O(i-1))) \\
&= (v(\{\sigma(1)\}) - v(\emptyset)) + (v(\{\sigma(1), \sigma(2)\}) - v(\{\sigma(1)\})) + \cdots + \\
&\quad (v(\{\sigma(1), \ldots, \sigma(k)\}) - v(\{\sigma(1), \ldots, \sigma(k-1)\})) + 0 \\
&= v(\{\sigma(1), \ldots, \sigma(k)\}) - v(\emptyset) \\
&= v(X).
\end{aligned}$$

By the same computation we have $U(f) = w(X)$ for every $X \subseteq \Omega$. Thus it must hold that $v(X) = w(X)$ for all $X \subseteq \Omega$. □

Proof of Proposition 9. Let $h := \alpha g + (1-\alpha)\langle p \rangle$ where $p \in \Delta(X)$ is such that $u(p) = 0$. Take $q \in \Delta(X)$ such that $\langle q \rangle \sim h$. Since we have Full Independence, $\frac{1}{2}f + \frac{1}{2}h \sim \frac{1}{2}f + \frac{1}{2}\langle q \rangle$. Again we use Lemma 3 to show that $\psi(F + \alpha G) = \psi(F) + \alpha\psi(G)$ for any $\alpha \in (0, 1)$. By homogeneity of ψ we have that $\psi(F + \alpha G) = \psi(F) + \alpha\psi(G)$ holds for any $\alpha > 0$ (same argument as in the proof of Proposition 7). □

A5. Proofs Section 6

Proof of Proposition 12. Fix a $\alpha \in (0, 1]$ and choose $f \in \mathcal{F}$ as well as $p, q \in \Delta(X)$ such that $p \succeq^* f(\omega) \succeq^* q$ for all $\omega \in \Omega$. Then for $s_0 \in \Delta(X)$ such that $u(s_0) = 0$ it holds by Axiom 5 that

$$\alpha p + (1-\alpha)s_0 \succeq^* \alpha f(\omega) + (1-\alpha)s_0 \succeq^* \alpha q + (1-\alpha)s_0,$$

for all $\omega \in \Omega$. Moreover, by Axiom 3 there exists a $\lambda \in [0, 1]$ such that

$$\lambda\left(\alpha\langle p \rangle + (1-\alpha)\langle s_0 \rangle\right) + (1-\lambda)\left(\alpha\langle q \rangle + (1-\alpha)\langle s_0 \rangle\right) \sim \alpha f + (1-\alpha)\langle s_0 \rangle.$$

Let $r := \lambda\langle p \rangle + (1-\lambda)\langle q \rangle$, then the above can be written as $\alpha r + (1-\alpha)\langle s_0 \rangle \sim \alpha f + (1-\alpha)\langle s_0 \rangle$. For every $k \in [-1, 1]$, let $s_k \in \Delta(X)$ be such that $u(s_k) = k$ and apply Axiom 11 to obtain $\alpha r + (1-\alpha)\langle s_k \rangle \sim \alpha f + (1-\alpha)\langle s_k \rangle$ and thus $\psi\left(u \circ (\alpha\langle r \rangle + (1-\alpha)\langle s_k \rangle)\right) = \psi\left(u \circ (\alpha f + (1-\alpha)\langle s_k \rangle)\right)$. Observe that

$$\begin{aligned}
\psi\left(u \circ (\alpha f + (1-\alpha)\langle s_k \rangle)\right) &= \psi\left(\alpha u \circ f + (1-\alpha)u \circ \langle s_k \rangle\right) \\
&= \psi\left(\alpha F + (1-\alpha)k\mathbb{1}_{\Omega}\right),
\end{aligned}$$

where $F = u \circ F$. Furthermore, note that

$$\begin{aligned}
\psi\left(u \circ (\alpha\langle r\rangle + (1-\alpha)\langle s_k\rangle)\right) &= U\left(\alpha\langle r\rangle + (1-\alpha)\langle s_k\rangle\right) \\
&= u\left(\alpha r + (1-\alpha)s_k\right) \\
&= \alpha u(r) + (1-\alpha)u(s_k) \\
&= \alpha u(r) + (1-\alpha)u(s_0) + (1-\alpha)u(s_k) \\
&= u(\alpha r + (1-\alpha)s_0) + (1-\alpha)k \\
&= U(\alpha\langle r\rangle + (1-\alpha)\langle s_0\rangle) + (1-\alpha)k \\
&= U(\alpha f + (1-\alpha)\langle s_0\rangle) + (1-\alpha)k \\
&= \psi(\alpha F + (1-\alpha)0\mathbb{1}_\Omega) + (1-\alpha)k \\
&= \psi(\alpha F) + (1-\alpha)k.
\end{aligned}$$

Thus we have $\psi(\alpha F) + (1-\alpha)k = \psi\left(u \circ (\alpha r + (1-\alpha)\langle s_k\rangle)\right) = \psi\left(u \circ (\alpha f + (1-\alpha)\langle s_k\rangle)\right) = \psi\left(\alpha F + (1-\alpha)k\mathbb{1}_\Omega\right)$.

Now suppose we have $\psi(F + \alpha\mathbb{1}_\Omega)$ for some $F \in [-1,1]^\Omega$ and $\alpha \in (0,1]$ such that $F + \alpha\mathbb{1}_\Omega \in [-1,1]^\Omega$. Then we have $\psi(F+\alpha\mathbb{1}_\Omega) = \psi\left((1-\alpha)\left(\frac{1}{(1-\alpha)}F\right) + \alpha\mathbb{1}_\Omega\right) = \psi\left((1-\alpha)\left(\frac{1}{(1-\alpha)}F\right)\right) + \alpha = \psi\left(F\right) + \alpha$. Thus we have shown that $\psi(F + \alpha\mathbb{1}_\Omega) = \psi\left(F\right) + \alpha$ for all $\alpha \in (0,1]$. Consider $\alpha > 1$ and let $F \in [-1,1]^\Omega$ then we have to show that $\psi(F + \alpha\mathbb{1}_\Omega) = \psi(F) + \alpha$ for $F + \alpha\mathbb{1}_\Omega \in [-1,1]^\Omega$. Note that $\alpha \leq 2$ since $\psi \in [-1,1]^\Omega$. Define $G := F + \mathbb{1}_\Omega$ and thus

$$\begin{aligned}
\psi(F + \alpha\mathbb{1}_\Omega) &= \psi(F + \mathbb{1}_\Omega + \alpha\mathbb{1}_\Omega - \mathbb{1}_\Omega) \\
&= \psi(G + (\alpha-1)\mathbb{1}_\Omega),
\end{aligned}$$

where $(\alpha - 1) < 1$. Therefore, it holds that $\psi(G+(\alpha-1)\mathbb{1}_\Omega) = \psi(G)+(\alpha-1) = \psi(F+\mathbb{1}_\Omega)+(\alpha-1) = \psi(F) + 1 + (\alpha-1) = \psi(F) + \alpha$ for all $\alpha > 1$. □

Proof of Theorem 7. First we show that $U(f) = \nabla\psi(\hat{F})F$, where $\hat{F} = F - sF$ for some $s \in (0,1)$. Take some act $f \in \mathcal{F}$ and $F \in [-1,1]^\Omega$ such that $u \circ f = F$ and note that $\nabla\psi(\hat{F})F = \psi^{'}(F - sF, F)$ by Lemma 1. Now take $F, (-F) \in [-1,1]^\Omega$ and since $(F - F) \in [-1,1]^\Omega$, by Theorem 6 there exists $s \in (0,1)$ such that $\psi(F - F) - \psi(F) = \psi^{'}(F - sF, -F)$. By homogeneity of $\psi^{'}$ in the second component it follows that $\psi^{'}(F - sF, -F) = -\psi^{'}(F - sF, F)$. Since $\psi(F - F) = \psi(0) = 0$ we have that $\psi(F) = \psi^{'}(F - sF, F)$ and therefore $U(f) = \psi^{'}(F - sF, F)$ and consequently $U(f) = \nabla\psi(\hat{F})F$ as shown above. □

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