Unawareness

- Minicourse -

Burkhard C. Schipper
University of California, Davis

Outline

1. Informal introduction
2. Epistemic models of unawareness
3. Type spaces with unawareness
4. Speculation
5. Bayesian games with unawareness
6. Revealed unawareness
7. Dynamic games with unawareness
8. …(???)

1. An informal introduction

“Awareness” in Natural Language

“I was aware of the red traffic light.” (Just knowledge?)

“Be of aware of sexually transmitted diseases!"

(“generally taking into account", “being present in mind", “paying attention to“)

Etymology: “aware” ← “wary” ← “gewær” (old English) ← “gewahr” (German)

Psychiatry: Lack of self-awareness means that a patient is oblivious to aspects of an illness that is obvious to his/her social contacts. (Failure of negative introspection.)

“Reports that say that something hasn't happened are always interesting to me, because as we know, there are known knowns; there are things we know we know. We also know there are known unknowns; that is to say we know there are some things we do not know. But there are also unknown unknowns – the ones we don't know we don't know. And if one looks throughout the history of our country and other free countries, it is the latter category that tend to be the difficult ones.”

(Former) United States Secretary of Defense, Donald Rumsfeld,
February 12, 2002

Unawareness as Lack of Conception

In most formal approaches:

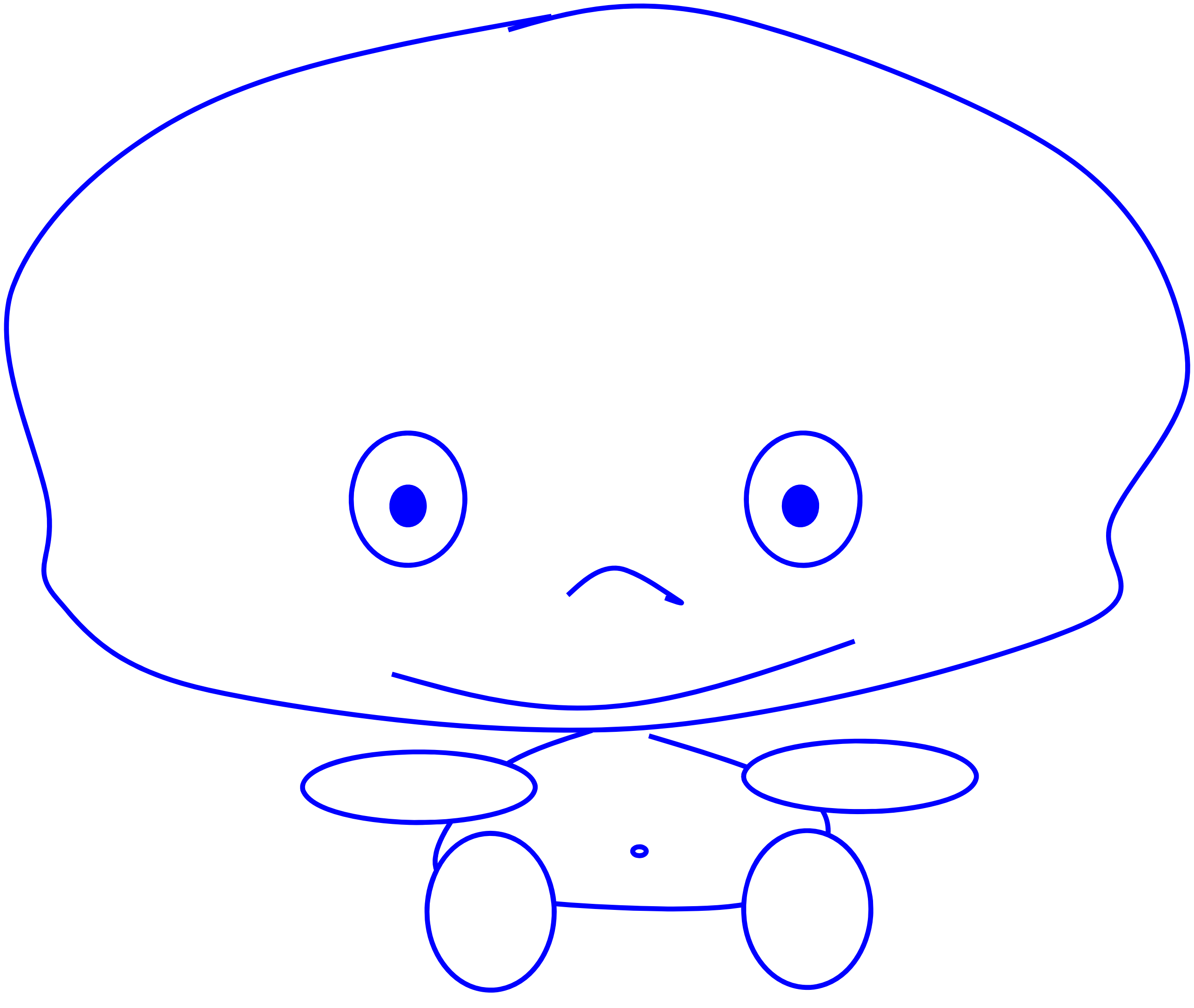
Unawareness means the lack of conception.

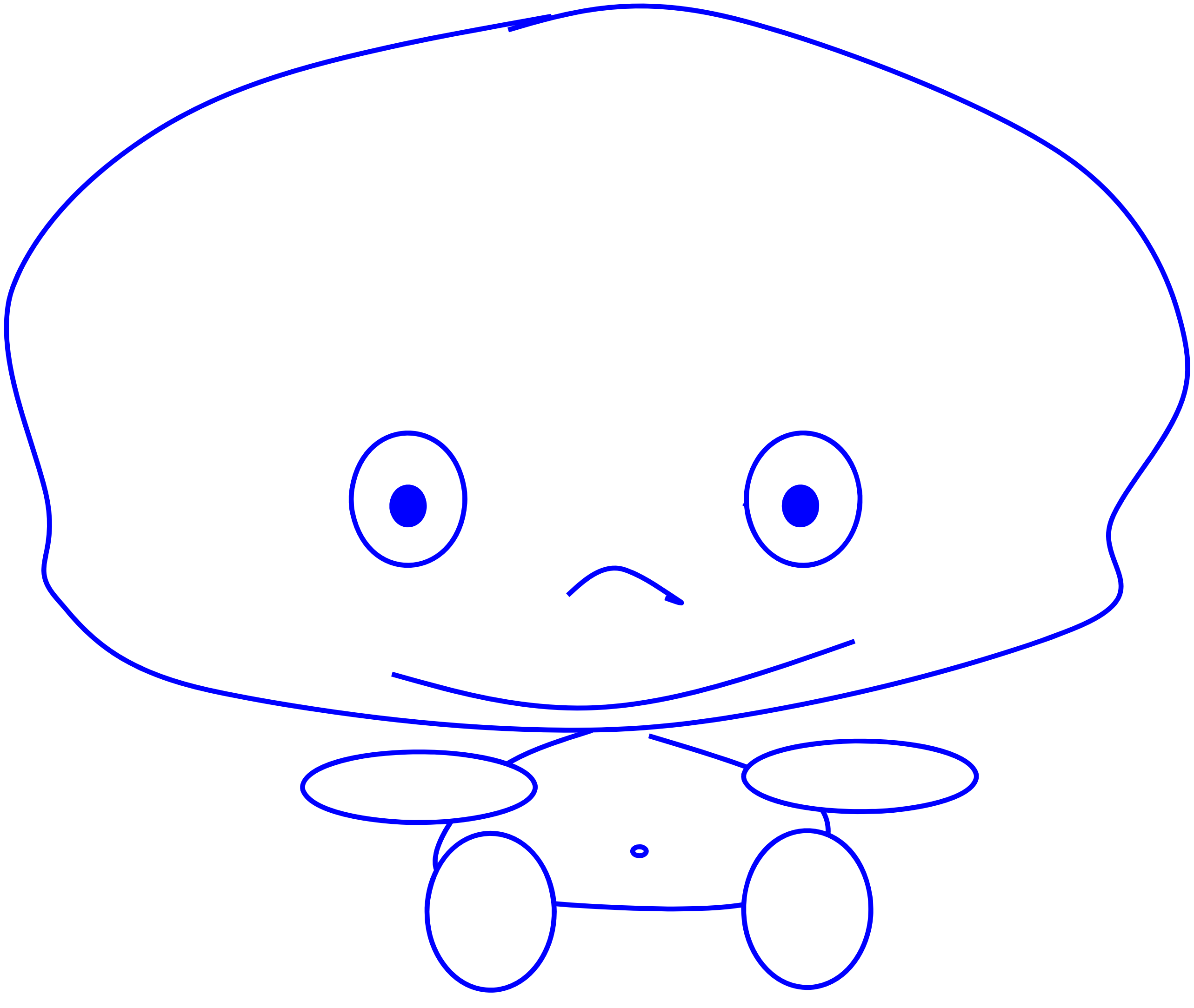
(“not being present in the mind”)

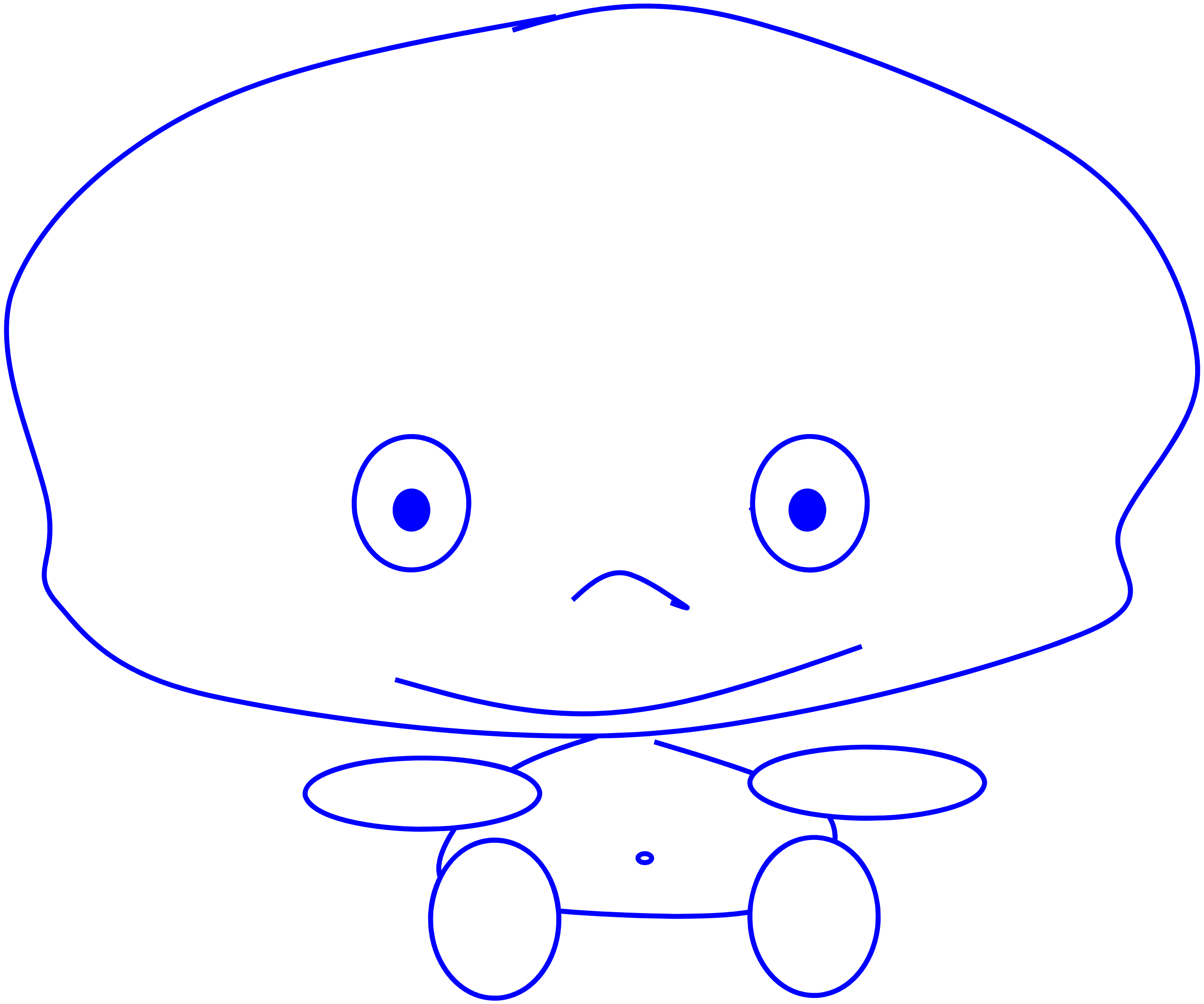
Lack of information versus lack of conception:

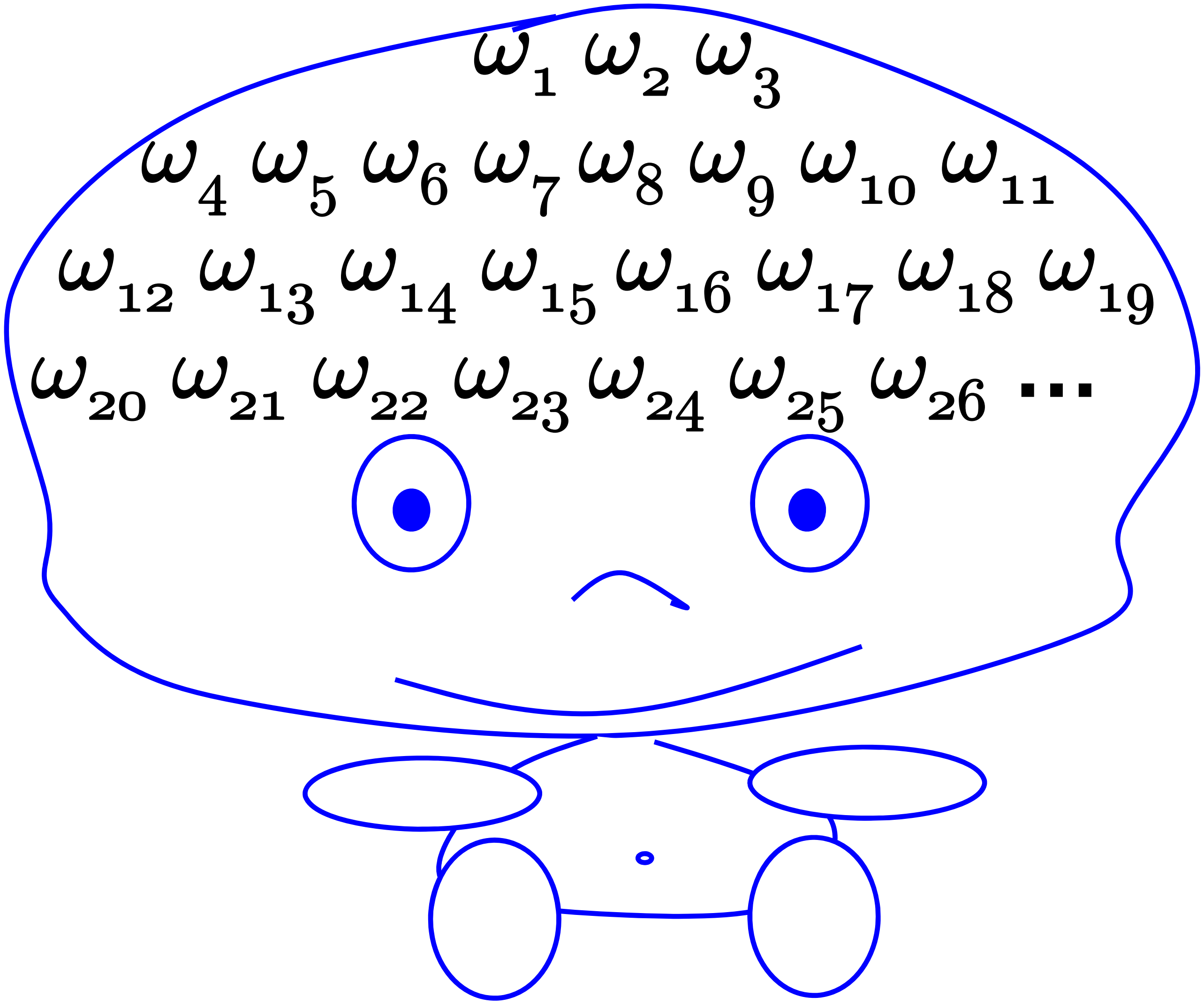
We all know the homo oeconomicus.

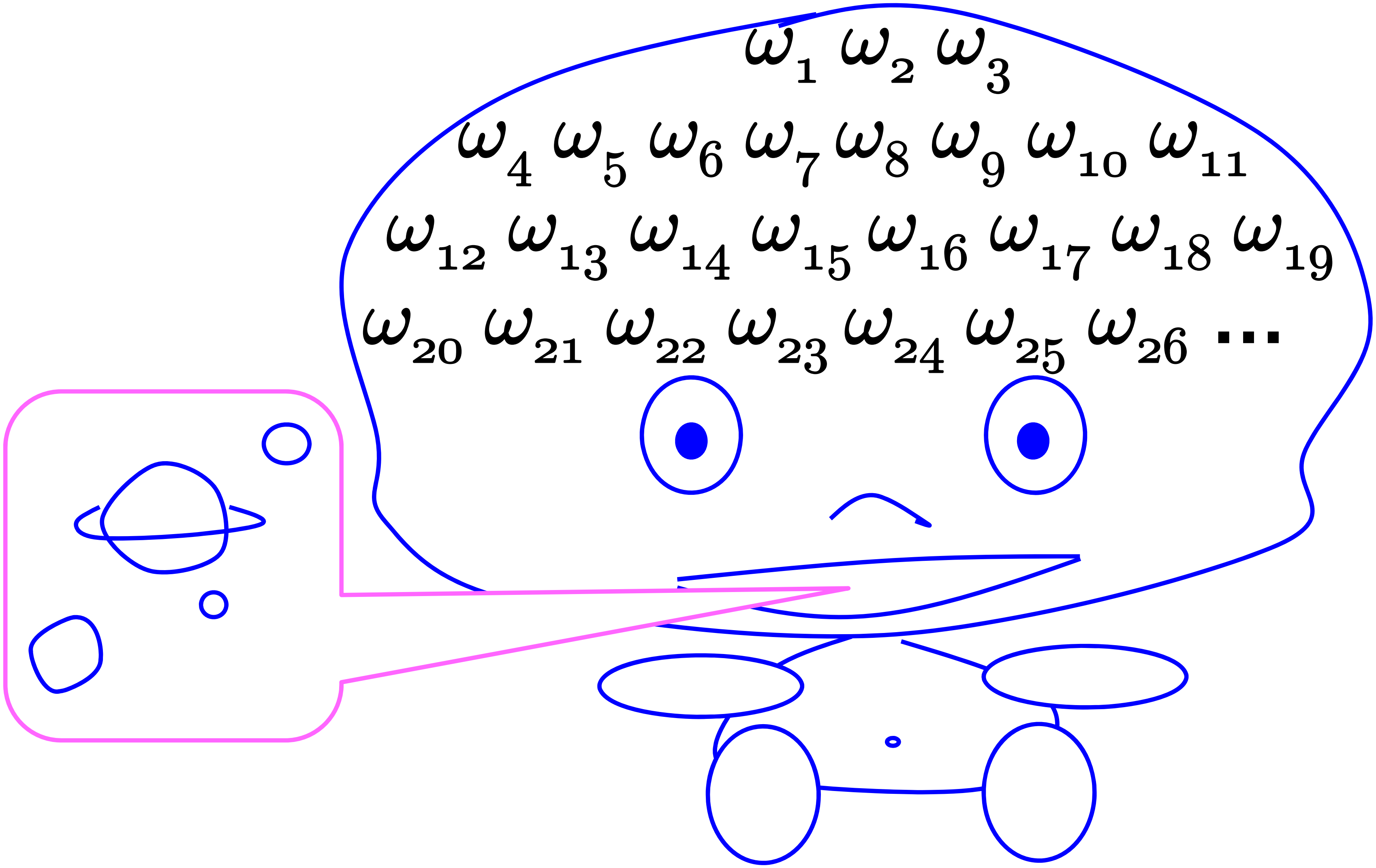
But how did he look as a baby?

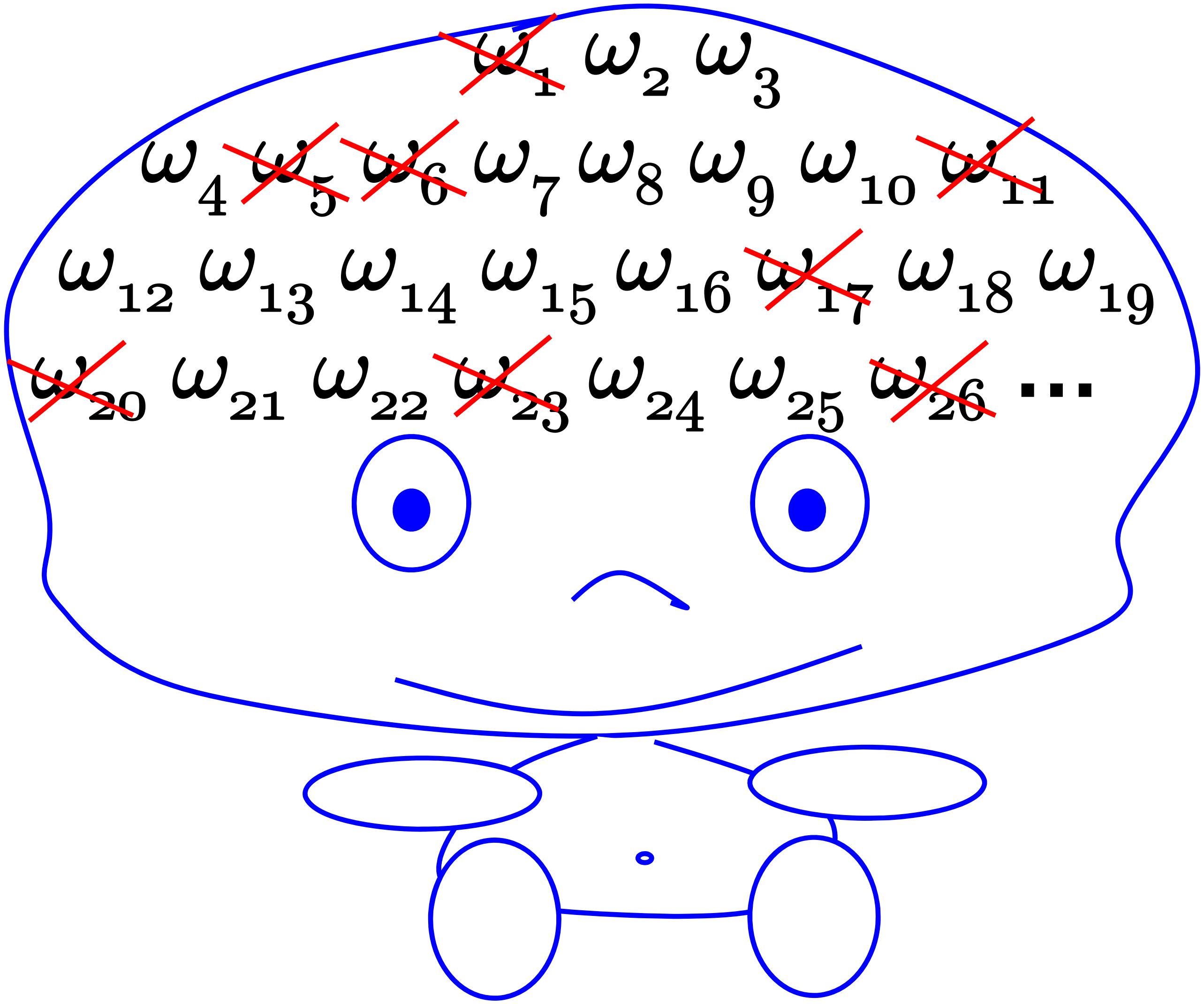







ω_1 ω_2 ω_3
ω_4 ω_5 ω_6 ω_7 ω_8 ω_9 ω_{10} ω_{11}
ω_{12} ω_{13} ω_{14} ω_{15} ω_{16} ω_{17} ω_{18} ω_{19}
ω_{20} ω_{21} ω_{22} ω_{23} ω_{24} ω_{25} ω_{26} ...


ω_1 ω_2 ω_3
ω_4 ω_5 ω_6 ω_7 ω_8 ω_9 ω_{10} ω_{11}
ω_{12} ω_{13} ω_{14} ω_{15} ω_{16} ω_{17} ω_{18} ω_{19}
ω_{20} ω_{21} ω_{22} ω_{23} ω_{24} ω_{25} ω_{26} ...


$\cancel{\omega_1}$ ω_2 ω_3
ω_4 $\cancel{\omega_5}$ $\cancel{\omega_6}$ ω_7 ω_8 ω_9 ω_{10} $\cancel{\omega_{11}}$
ω_{12} ω_{13} ω_{14} ω_{15} ω_{16} $\cancel{\omega_{17}}$ ω_{18} ω_{19}
$\cancel{\omega_{20}}$ ω_{21} ω_{22} $\cancel{\omega_{23}}$ ω_{24} ω_{25} $\cancel{\omega_{26}}$ ...

Standard models allow for the lack of information but not for the lack of conception.

In standard models, learning means shrinking the relevant state space but never discovering of new possibilities.

An agent may have lack of conception of an event because:

- never thought about it (i.e., novelties)
- does not pay attention to it at the very moment we model (different from rational inattention)

Relevance of unawareness

- It is real phenomena
- Incomplete contracting, incomplete markets
- Speculation in financial markets
- Disclosure of information
- Strategic negotiations and bargaining
- Modelling discoveries and innovations
- Modeling games where the perception of the strategic context is not necessarily “common” among players
- Exploring robustness of decision theory to small changes in assumptions on the primitives
- …

Non-robustness of Decision Theory

In decision theory, the decision maker's perception of the problem are captured in the primitives (state space, set of consequences, acts etc.)

These primitives are assumed by the modeler and are not revealed.

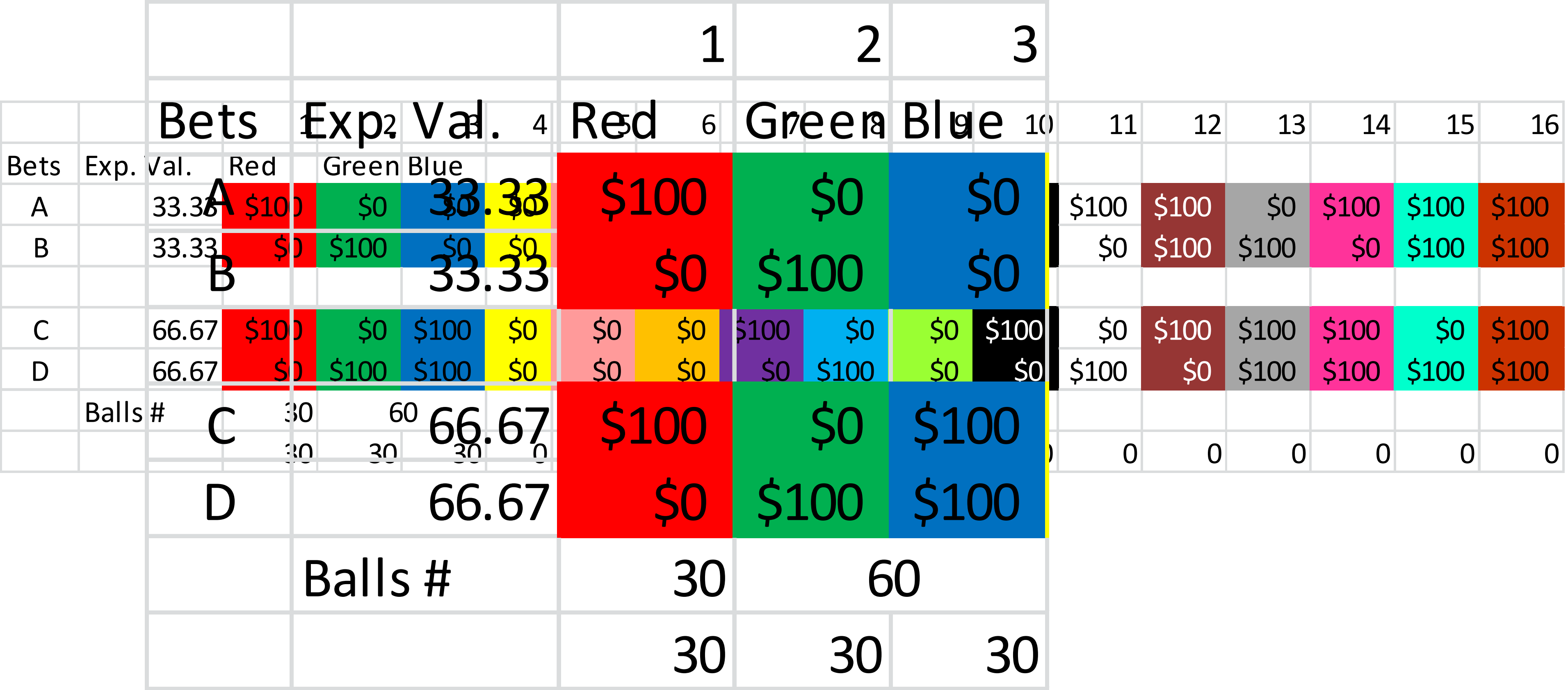
Preference are revealed *given* primitives.

How do we know that the decision maker views the problem the same as the modeler?

Example: Non-robustness of the Ellsberg Paradox

Bets		Red	Green	Blue
A		$100	$0	$0
B		$0	$100	$0
C		$100	$0	$100
D		$0	$100	$100
	Balls #	30	60	

Example: Non-robustness of the Ellsberg Paradox



		1	2	3
Bets	Exp. Val.	Red	Green	Blue
A	33.33	$100	$0	$0
B	33.33	$0	$100	$0
C	66.67	$100	$0	$100
D	66.67	$0	$100	$100
	Balls #	30	60	
		30	30	30

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Bets	Exp. Val.	Red	Green	Blue													
A	33.33	$100	$0	$0	$0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	$100	$100	$0	$100	$100	$100
B	33.33	$0	$100	$0	$0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	$0	$100	$100	$0	$100	$100
C	66.67	$100	$0	$100	$0	$0	$0	$100	$0	$0	$100	$0	$100	$100	$100	$0	$100
D	66.67	$0	$100	$100	$0	$0	$0	$0	$100	$0	$0	$100	$0	$100	$100	$100	$100
	Balls #	30	60														
		30	30	30	0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	0	0	0	0	0	0

Example: Non-robustness of the Ellsberg Paradox

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Bets	Exp. Val.	Red	Green	Blue													
A	34.07	$100	$0	$0	$0	$100	$0	$0	$0	$100	$0	$100	$100	$0	$100	$100	$100
B	32.97	$0	$100	$0	$0	$0	$100	$0	$0	$100	$100	$0	$100	$100	$0	$100	$100
C	65.93	$100	$0	$100	$0	$0	$0	$100	$0	$0	$100	$0	$100	$100	$100	$0	$100
D	67.03	$0	$100	$100	$0	$0	$0	$0	$100	$0	$0	$100	$0	$100	$100	$100	$100
	Balls #	30	60														
		30	30	30	0	0	0	0	0	0	0	1	0	0	0	0	0

Rationalizes Ellsberg behavior with expected utility on a slightly different state-space.

Unawareness models allow us to analyze decisions under varying primitives.

How to model unawareness?

2. Epistemic models of unawareness

What’s the problem with modeling unawareness?

Why not take a state-space model a la Aumann or a Kripke frame?

What's the problem with modeling unawareness?

a state space S

a knowledge operator

$\mathbf{K}_i : 2^S \to 2^S$

an unawareness operator

$\mathbf{U}_i : 2^S \to 2^S$

Properties:

$\mathbf{K}_i(S) = S$ (Necessitation)

Digression: Necessitation of Belief

Digression: Necessitation of Belief

Digression: Necessitation of Belief

What's the problem with modeling unawareness?

a state space S

a knowledge operator

$\mathbf{K}_i : 2^S \to 2^S$

an unawareness operator

$\mathbf{U}_i : 2^S \to 2^S$

Properties:

$\mathbf{K}_i(S) = S$ (Necessitation)

$\mathbf{U}_i(E) \subseteq \neg\mathbf{K}_i(E) \cap \neg\mathbf{K}_i\neg\mathbf{K}_i(E)$ (Plausibility)

$\mathbf{K}_i\mathbf{U}_i(E) \subseteq \emptyset$ (KU Introspection)

$\mathbf{U}_i(E) \subseteq \mathbf{U}_i\mathbf{U}_i(E)$ (AU Reflection)

Observation 1 (Dekel, Lipman, and Rustichini, 1998) *If a state-space model satisfies Plausibility, KU-introspection, AU-reflection, and Necessitation, then* $\mathbf{U}_i(E) = \emptyset$*, for any event* $E \in 2^S$.

PROOF. $\mathbf{U}_i(E) \overset{AU-Refl.}{\subseteq} \mathbf{U}_i(\mathbf{U}_i(E)) \overset{Plaus.}{\subseteq} \neg\mathbf{K}_i(\neg\mathbf{K}_i(\mathbf{U}_i(E))) \overset{KU-Intro.}{=} \neg\mathbf{K}_i(\Omega) \overset{Nec.}{=} \emptyset.$ $\square$

Modeling Unawareness

Various approaches, interdisciplinary

Computer science

- Modeling logical non-omniscience and limited reasoning
- Inspired by Kripke structures
- Analyst’s description of agents’ reasoning
- Seminal work: Fagin and Halpern (AI 1988)

Economics

- Focused on modeling lack of conception while keeping everything else standard
- Inspired by Aumann structures and Harsanyi type spaces
- Players’ descriptions of players’ reasoning

Unawareness Structures

Goals:

1. Define a structure consistent with non-trivial unawareness in the multi-agent case.
2. Prove all properties of awareness of Dekel, Lipman, Rustichini (1998), Modica and Rustichini (1999), Halpern (2001)
3. Prove unawareness is consistent with strong notions of knowledge (like S4 or stronger).
4. Clear separation between syntax and semantics to facilitate applications.

$\langle \mathcal{S}, \preceq \rangle$ nonempty complete lattice of nonempty disjoint spaces

Digression: Lattices

Let X is a set and $\preceq$ be a binary relation defined on X.

$\preceq$ is *partial order* if it is

reflexive: $x \preceq x$ for all $x \in X$

antisymmetric: for any $x, y \in X$, $x \preceq y$ and $y \preceq x$ implies $x = y$

transitive: for any $x, y, z \in X$, $x \preceq y$ and $y \preceq z$ implies $x \preceq z$

Digression: Lattices

Let $Y \subseteq X$.

u is an *upper bound* of Y if $x \preceq u$ for all $x \in Y$

ℓ is a *lower bound* of Y if $\ell \preceq x$ for all $x \in Y$

u is a *least upper bound, join, or surpremum* of Y if $u \preceq x$ for all upper bounds x of Y

ℓ is a *greatest lower bound, meet, or infimum* of Y if $x \preceq \ell$ for all lower bounds x of Y

Digression: Lattices

X is a *lattice* if for any $x, y \in X$ has a join and meet.

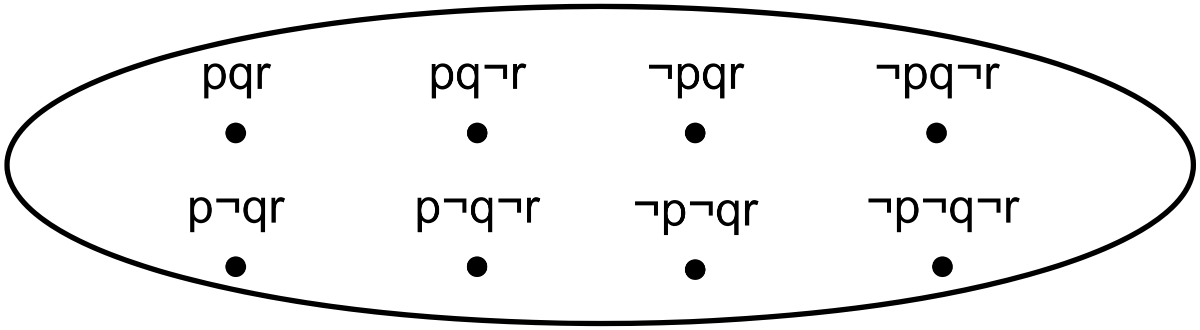
X is a *complete lattice* if any $Y \subseteq X$ has a join and a meet.

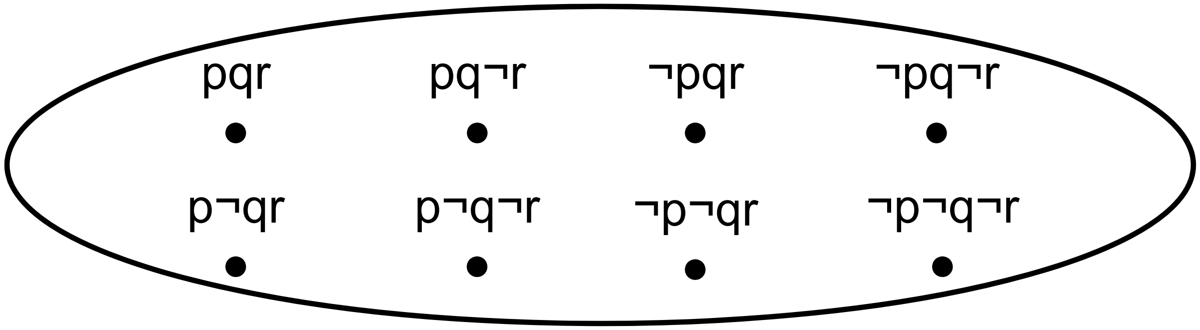
Every nonempty finite lattice is complete.

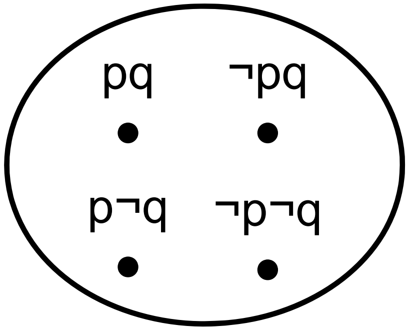
For any set, the set of all subsets is a complete lattice ordered by set inclusion.

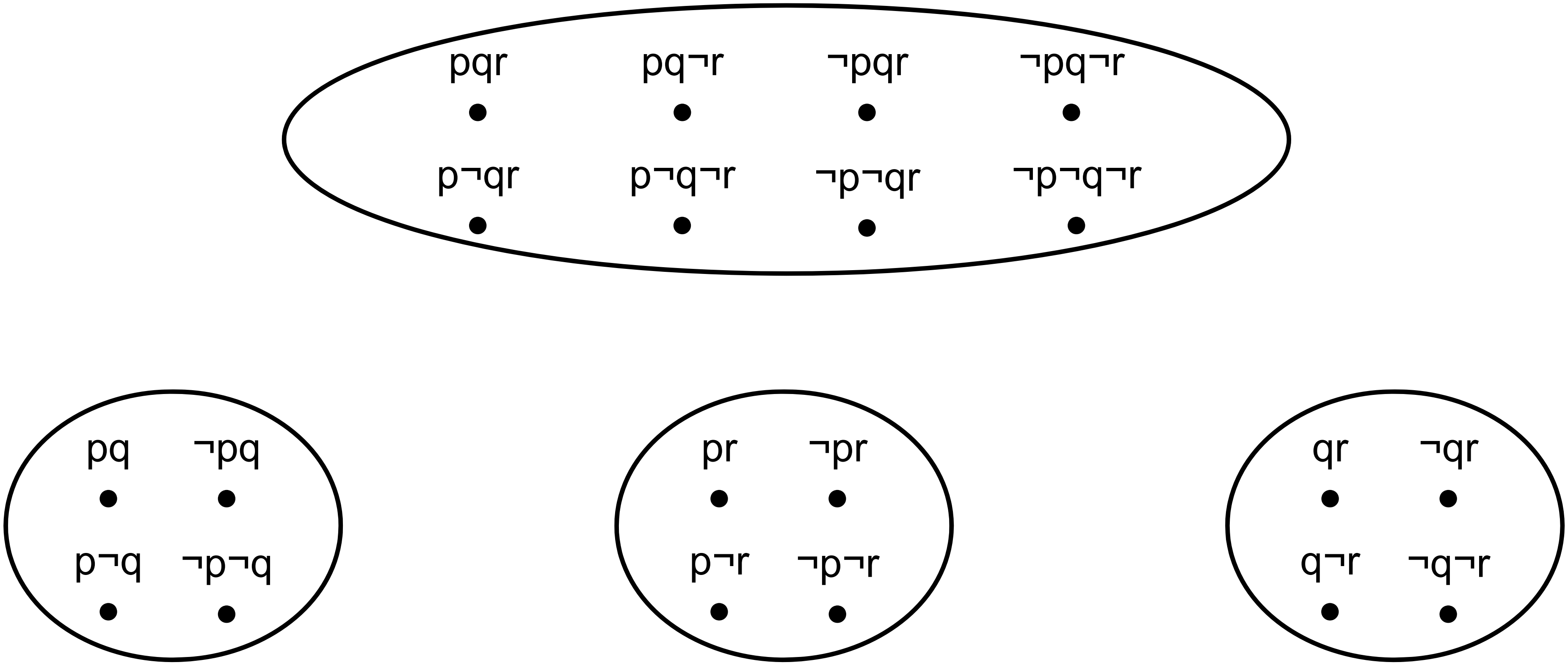
$\langle \mathcal{S}, \preceq \rangle$ nonempty complete lattice of nonempty disjoint spaces

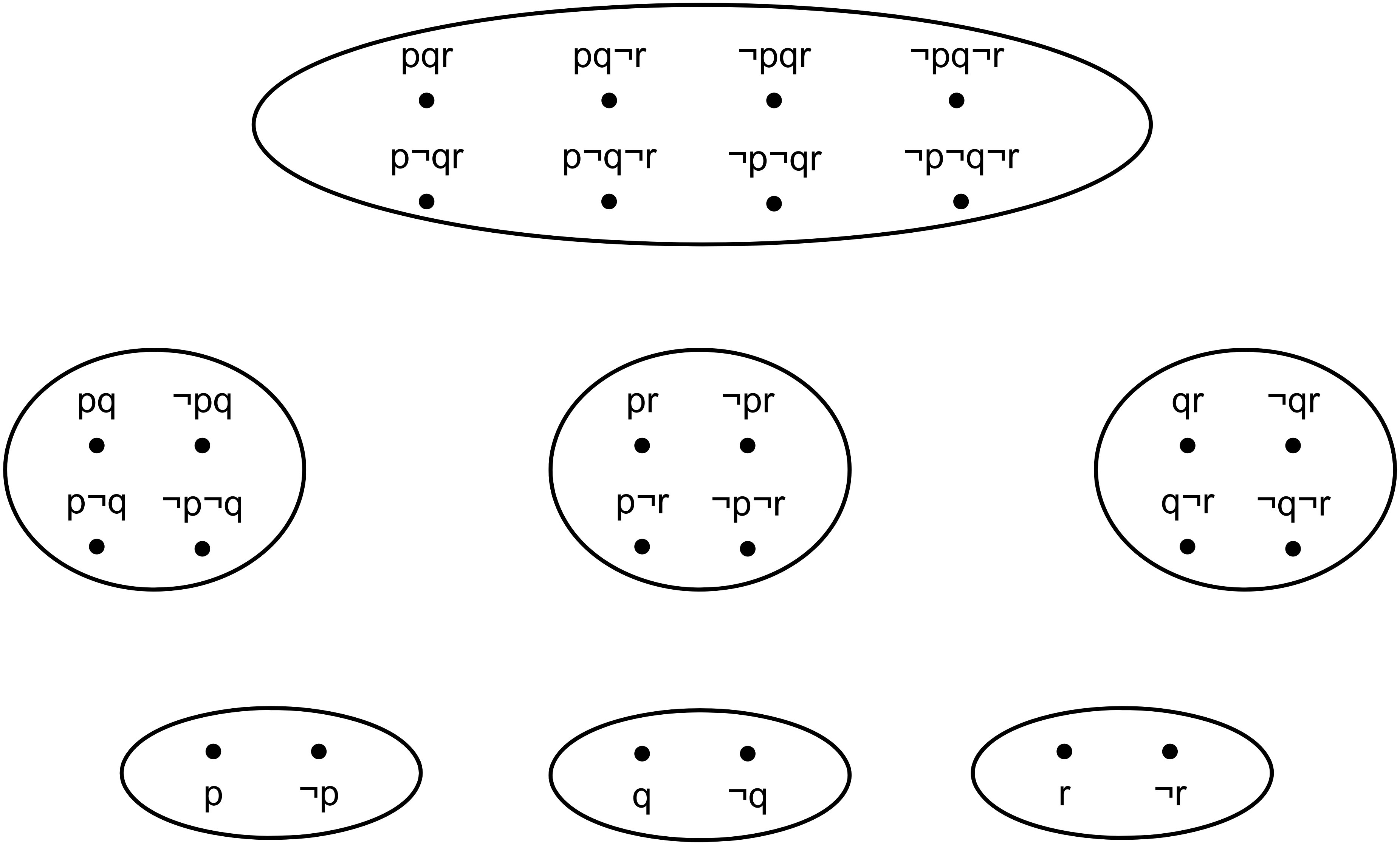
For $S, S' \in \mathcal{S}$, $S' \succeq S$ stands for "S' is more expressive than S"

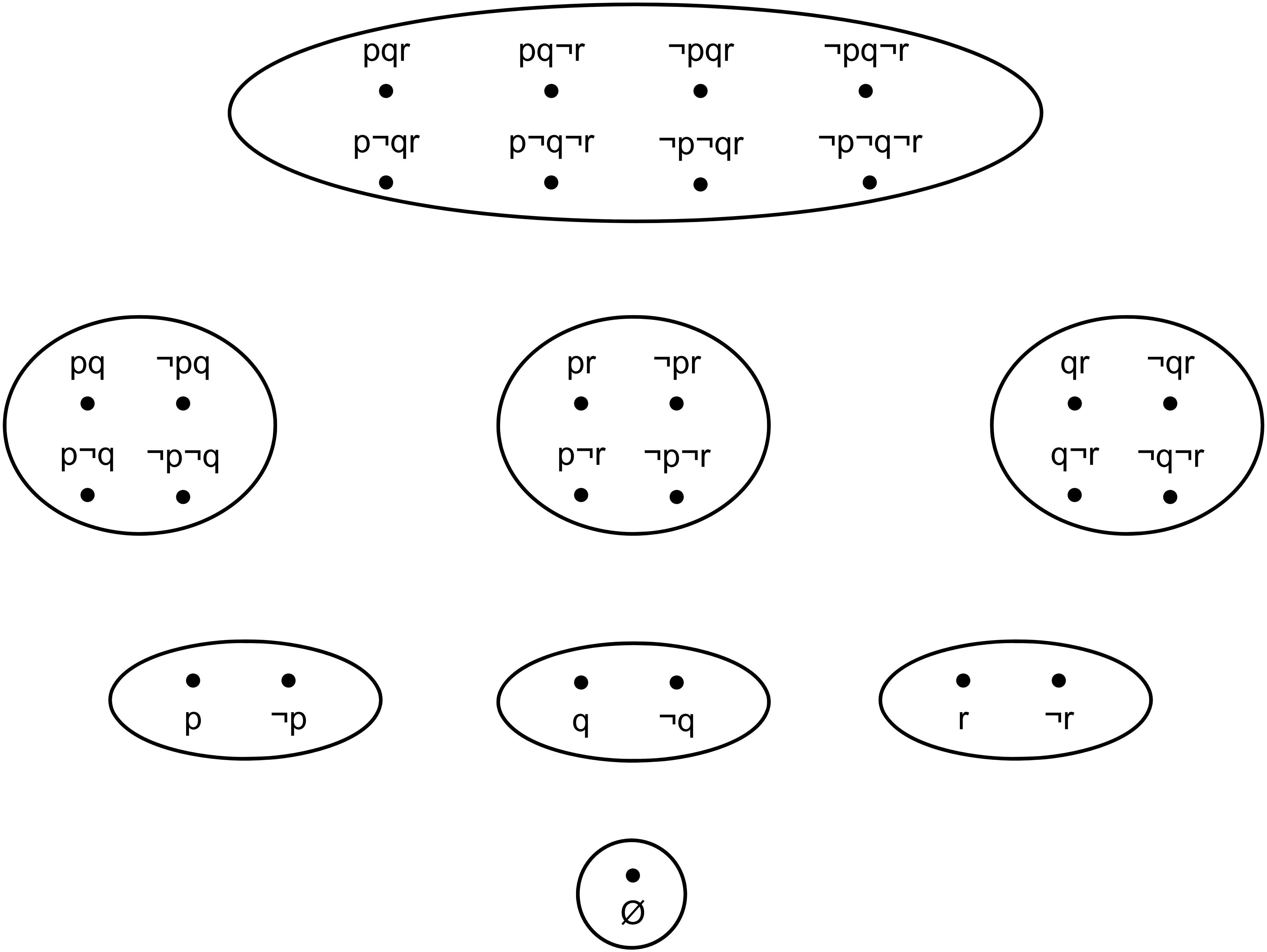

pqr
pq¬r
¬pqr
¬pq¬r
p¬qr
p¬q¬r
¬p¬qr
¬p¬q¬r


pqr
pq¬r
¬pqr
¬pq¬r
p¬qr
p¬q¬r
¬p¬qr
¬p¬q¬r


pq
¬pq
p¬q
¬p¬q


pqr
pq¬r
¬pqr
¬pq¬r
p¬qr
p¬q¬r
¬p¬qr
¬p¬q¬r
pq
¬pq
p¬q
¬p¬q
pr
¬pr
p¬r
¬p¬r
qr
¬qr
q¬r
¬q¬r


pqr
pq¬r
¬pqr
¬pq¬r
p¬qr
p¬q¬r
¬p¬qr
¬p¬q¬r
pq
¬pq
p¬q
¬p¬q
pr
¬pr
p¬r
¬p¬r
qr
¬qr
q¬r
¬q¬r
p
¬p
q
¬q
r
¬r


pqr
pq¬r
¬pqr
¬pq¬r
p¬qr
p¬q¬r
¬p¬qr
¬p¬q¬r
pq
¬pq
p¬q
¬p¬q
pr
¬pr
p¬r
¬p¬r
qr
¬qr
q¬r
¬q¬r
p
¬p
q
¬q
r
¬r
Ø

$\langle \mathcal{S}, \preceq \rangle$ nonempty complete lattice of nonempty disjoint spaces

For $S, S' \in \mathcal{S}$, $S' \succeq S$ stands for "S' is more expressive than S"

$\Omega := \bigcup_{S \in \mathcal{S}} S$

$\langle \mathcal{S}, \preceq \rangle$ nonempty complete lattice of nonempty disjoint spaces

For $S, S' \in \mathcal{S}$, $S' \succeq S$ stands for "S' is more expressive than S"

$\Omega := \bigcup_{S \in \mathcal{S}} S$

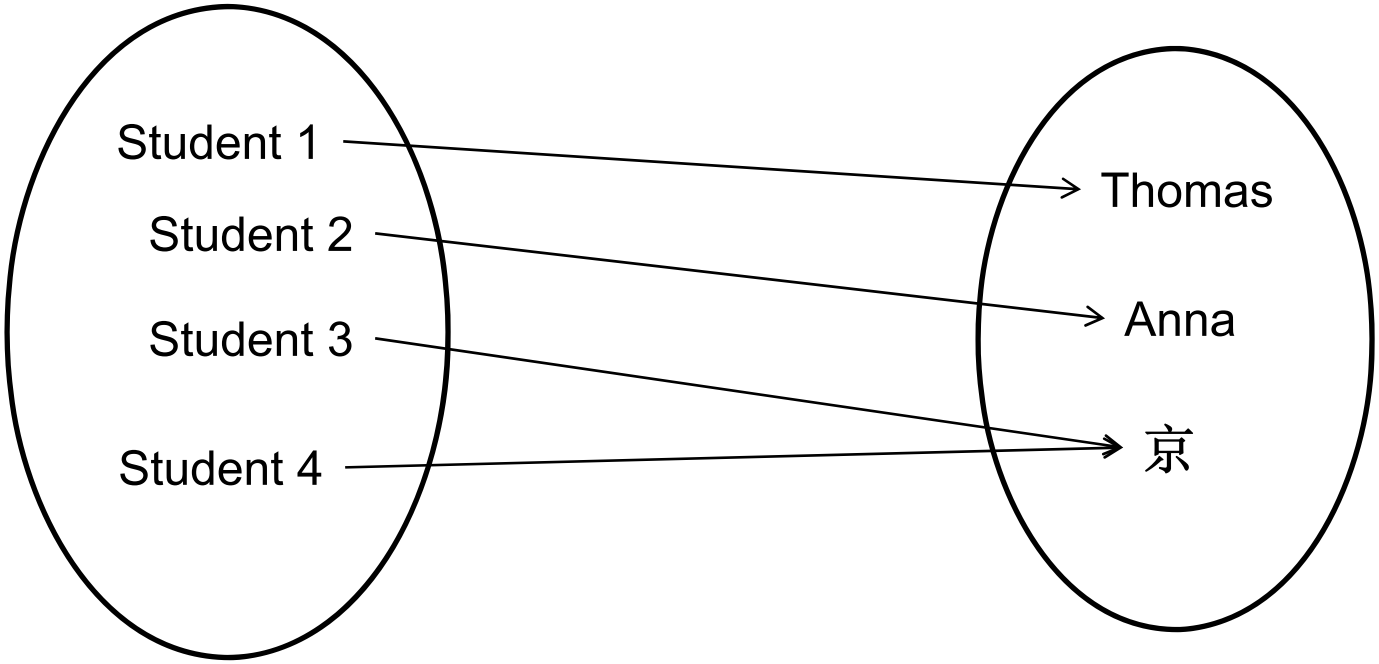
For $S, S' \in \mathcal{S}$ with $S' \succeq S$, $r_S^{S'} : S' \longrightarrow S$ surjective projection.

For any $S \in \mathcal{S}$, $r_S^S = id_S$.

For any $S, S', S'' \in \mathcal{S}$, $S'' \succeq S' \succeq S$, $r_S^{S''} = r_S^{S'} \circ r_{S'}^{S''}$.

Digression: Surjection

A function $f : X \longrightarrow Y$ is surjective or onto if for every $y \in Y$ there exists an $x \in X$ such that $f(x) = y$.



$\langle \mathcal{S}, \preceq \rangle$ nonempty complete lattice of nonempty disjoint spaces

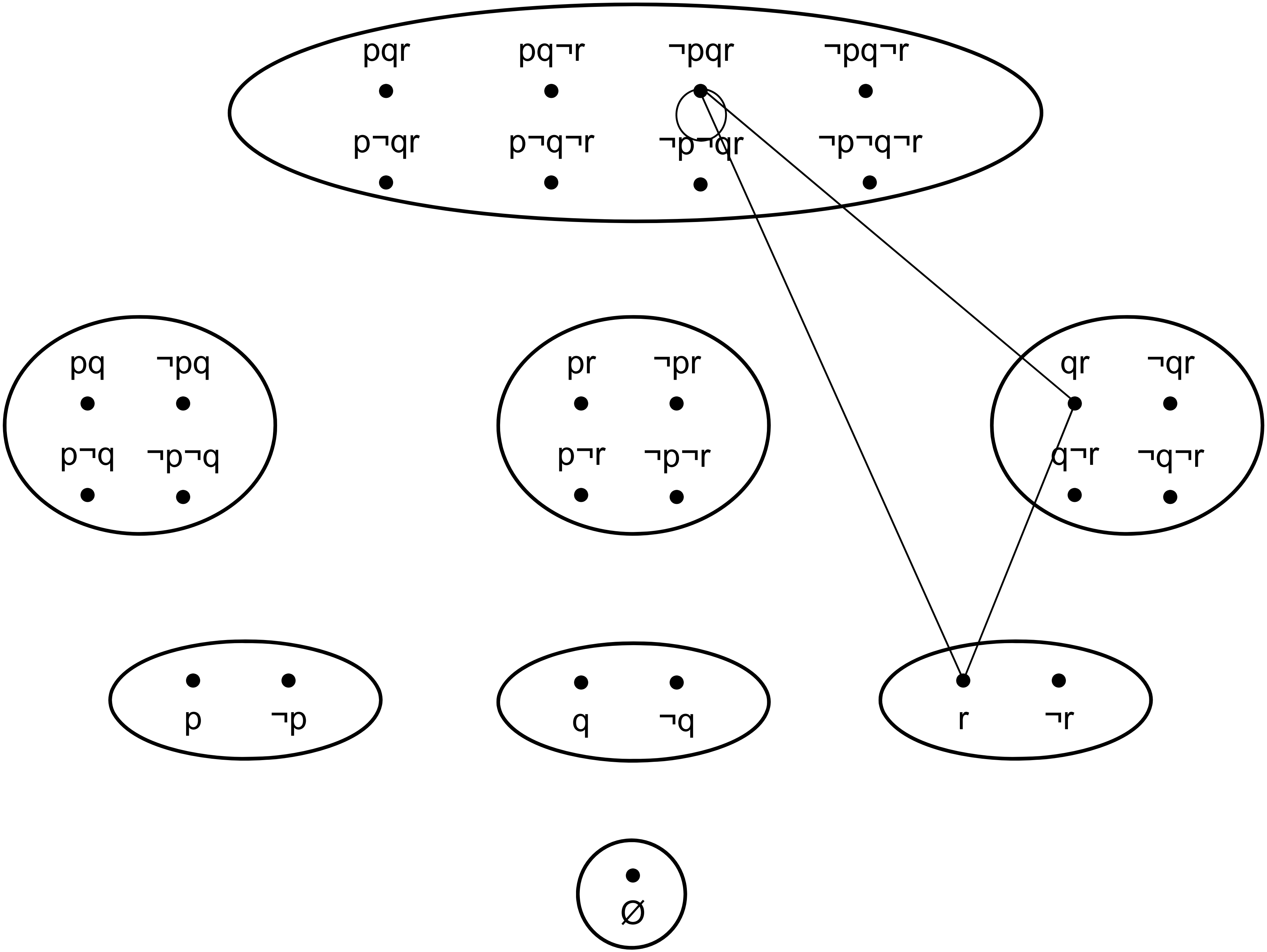
For $S, S' \in \mathcal{S}$, $S' \succeq S$ stands for "S' is more expressive than S"

$\Omega := \bigcup_{S \in \mathcal{S}} S$

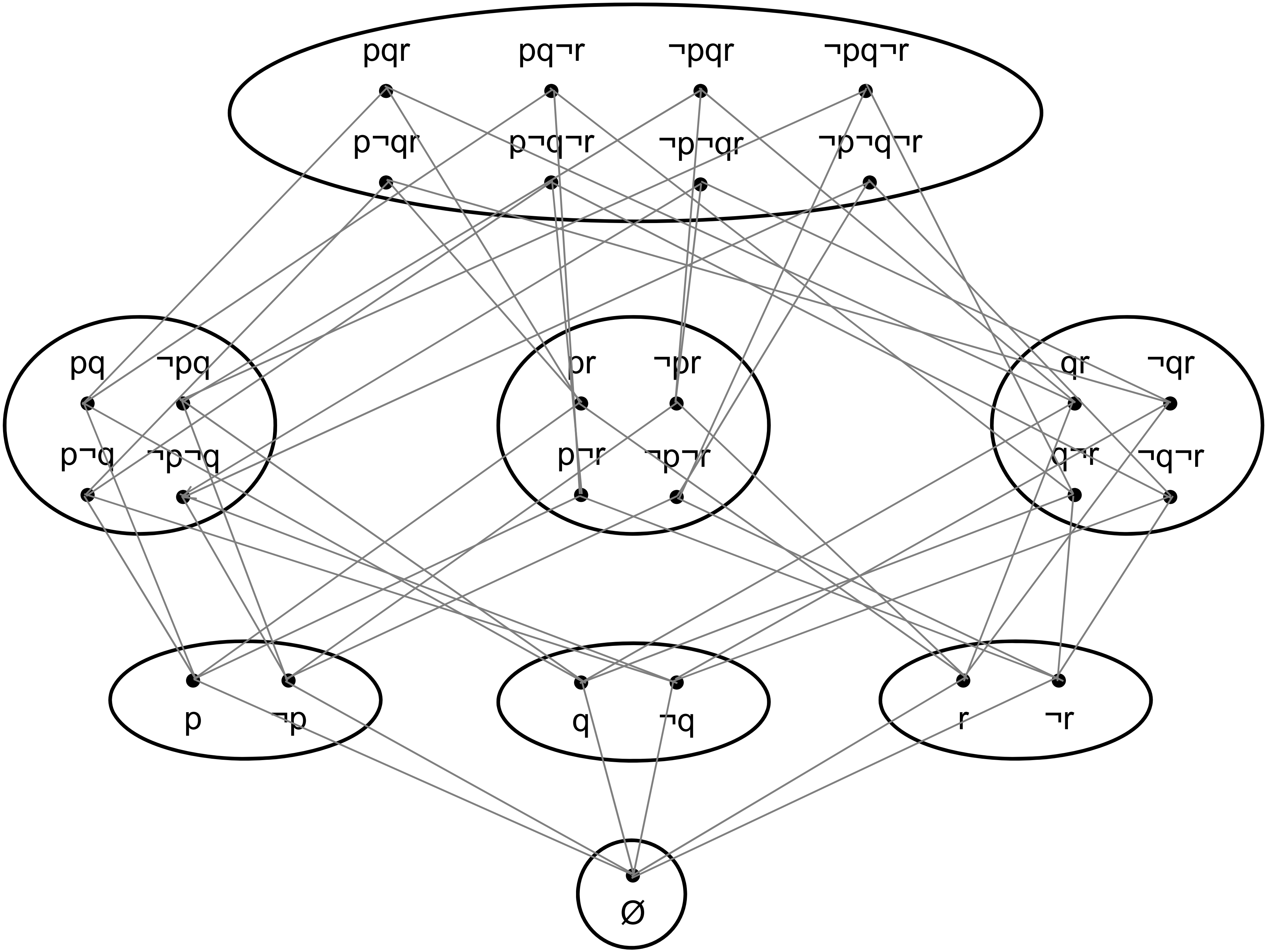
For $S, S' \in \mathcal{S}$ with $S' \succeq S$, $r_S^{S'} : S' \longrightarrow S$ surjective projection.

For any $S \in \mathcal{S}$, $r_S^S = id_S$.

For any $S, S', S'' \in \mathcal{S}$, $S'' \succeq S' \succeq S$, $r_S^{S''} = r_S^{S'} \circ r_{S'}^{S''}$.



pqr
pq¬r
¬pqr
¬pq¬r
p¬qr
p¬q¬r
¬p¬qr
¬p¬q¬r
pq
¬pq
p¬q
¬p¬q
pr
¬pr
p¬r
¬p¬r
qr
¬qr
q¬r
¬q¬r
p
¬p
q
¬q
r
¬r
Ø


pqr
pq¬r
¬pqr
¬pq¬r
p¬qr
p¬q¬r
¬p¬qr
¬p¬q¬r
pq
¬pq
p¬q
¬p¬q
pr
¬pr
p¬r
¬p¬r
qr
¬qr
q¬r
¬q¬r
p
¬p
q
¬q
r
¬r
Ø

$\langle \mathcal{S}, \preceq \rangle$ nonempty complete lattice of nonempty disjoint spaces

For $S, S' \in \mathcal{S}$, $S' \succeq S$ stands for "S' is more expressive than S"

$\Omega := \bigcup_{S \in \mathcal{S}} S$

For $S, S' \in \mathcal{S}$ with $S' \succeq S$, $r_S^{S'} : S' \longrightarrow S$ surjective projection.

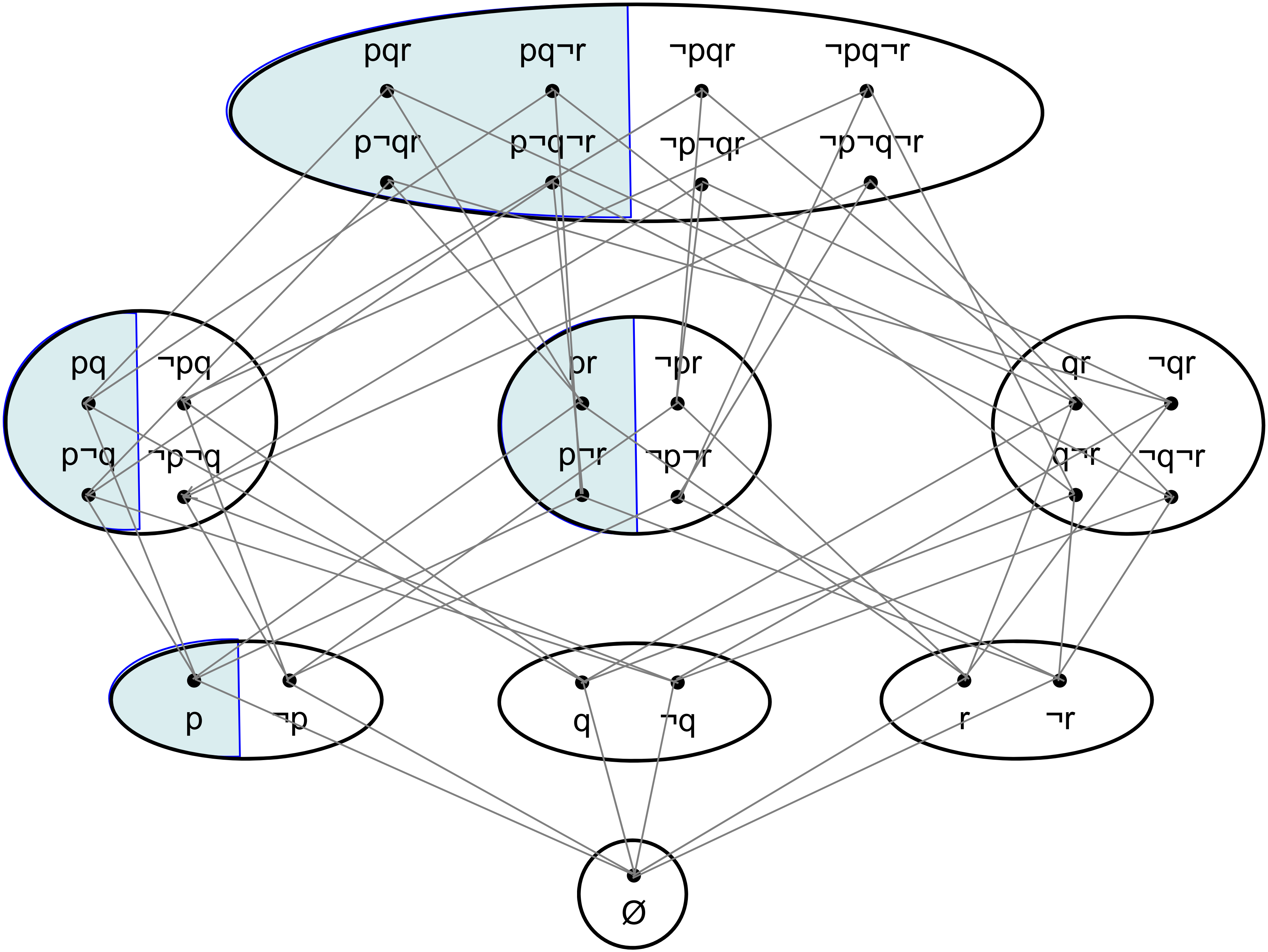
For any $S \in \mathcal{S}$, $r_S^S = id_S$.

For any $S, S', S'' \in \mathcal{S}$, $S'' \succeq S' \succeq S$, $r_S^{S''} = r_S^{S'} \circ r_{S'}^{S''}$.

For $\omega \in S'$, $\omega_S = r_S^{S'}(\omega)$. For $D \subset S'$, $D_S = \{\omega_S : \omega \in D\}$.

For $D \subseteq S$, $D^{\uparrow} := \bigcup_{S' \in \mathcal{S} : S' \succeq S} \left(r_S^{S'} \right)^{-1} (D)$.

$E \subseteq \Omega$ is an *event* if $E = D^{\uparrow}$ for some *base* $D \subseteq S$ in some *base space* $S \in \mathcal{S}$. $(S(E))$

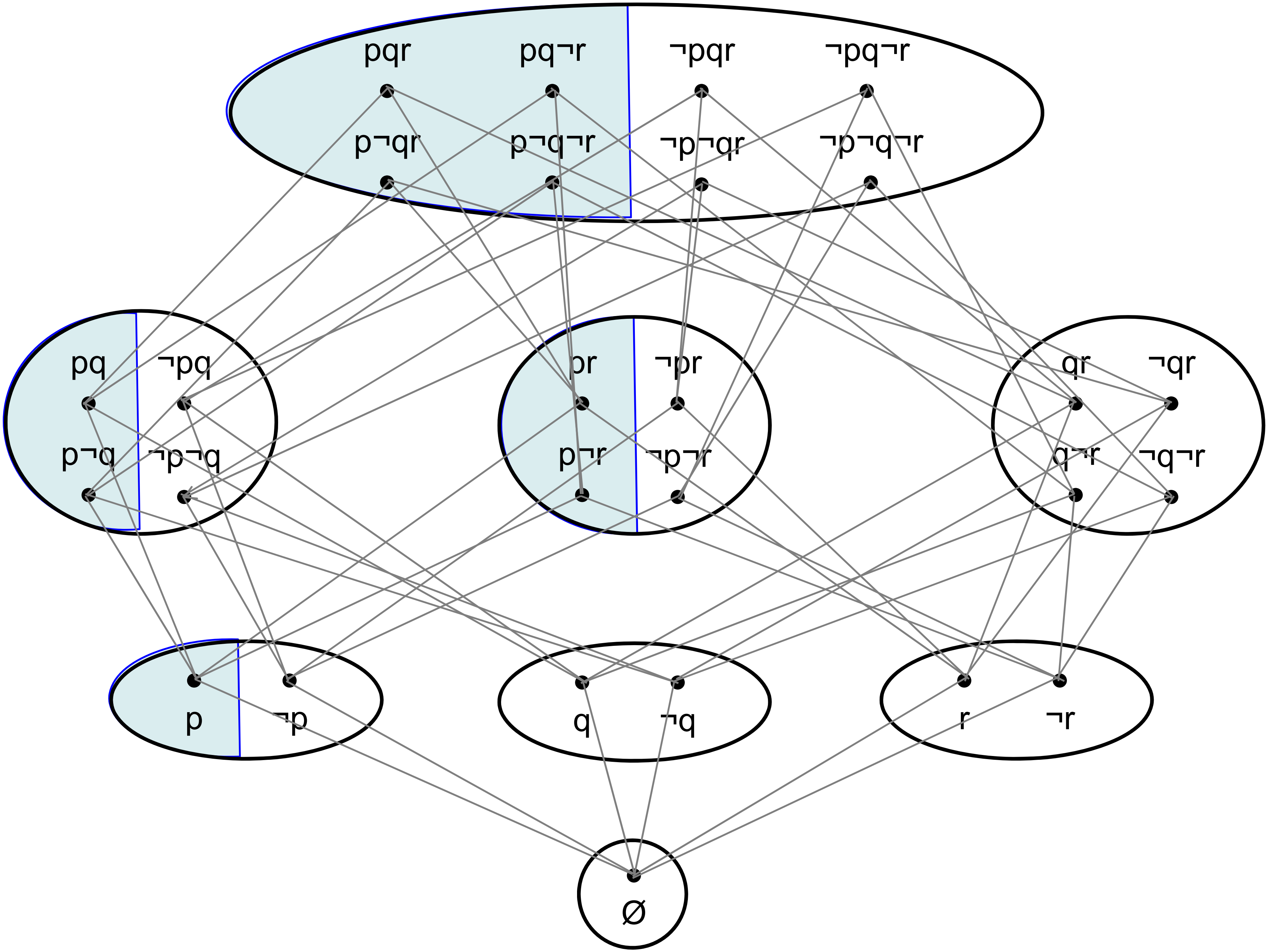

pqr
pq¬r
¬pqr
¬pq¬r
p¬qr
p¬q¬r
¬p¬qr
¬p¬q¬r
pq
¬pq
p¬q
¬p¬q
pr
¬pr
p¬r
¬p¬r
qr
¬qr
q¬r
¬q¬r
p
¬p
q
¬q
r
¬r
Ø

For $D \subseteq S$, $D^{\uparrow} := \bigcup_{S' \in \mathcal{S} : S' \succeq S} \left(r_S^{S'} \right)^{-1} (D)$.

$E \subseteq \Omega$ is an *event* if $E = D^{\uparrow}$ for some *base* $D \subseteq S$ in some *base space* $S \in \mathcal{S}$. $(S(E))$

We write $\emptyset^S$ for the vacuous event with base space S.

Not every subset of Ω is an event.


pqr
pq¬r
¬pqr
¬pq¬r
p¬qr
p¬q¬r
¬p¬qr
¬p¬q¬r
pq
¬pq
p¬q
¬p¬q
pr
¬pr
p¬r
¬p¬r
qr
¬qr
q¬r
¬q¬r
p
¬p
q
¬q
r
¬r
Ø

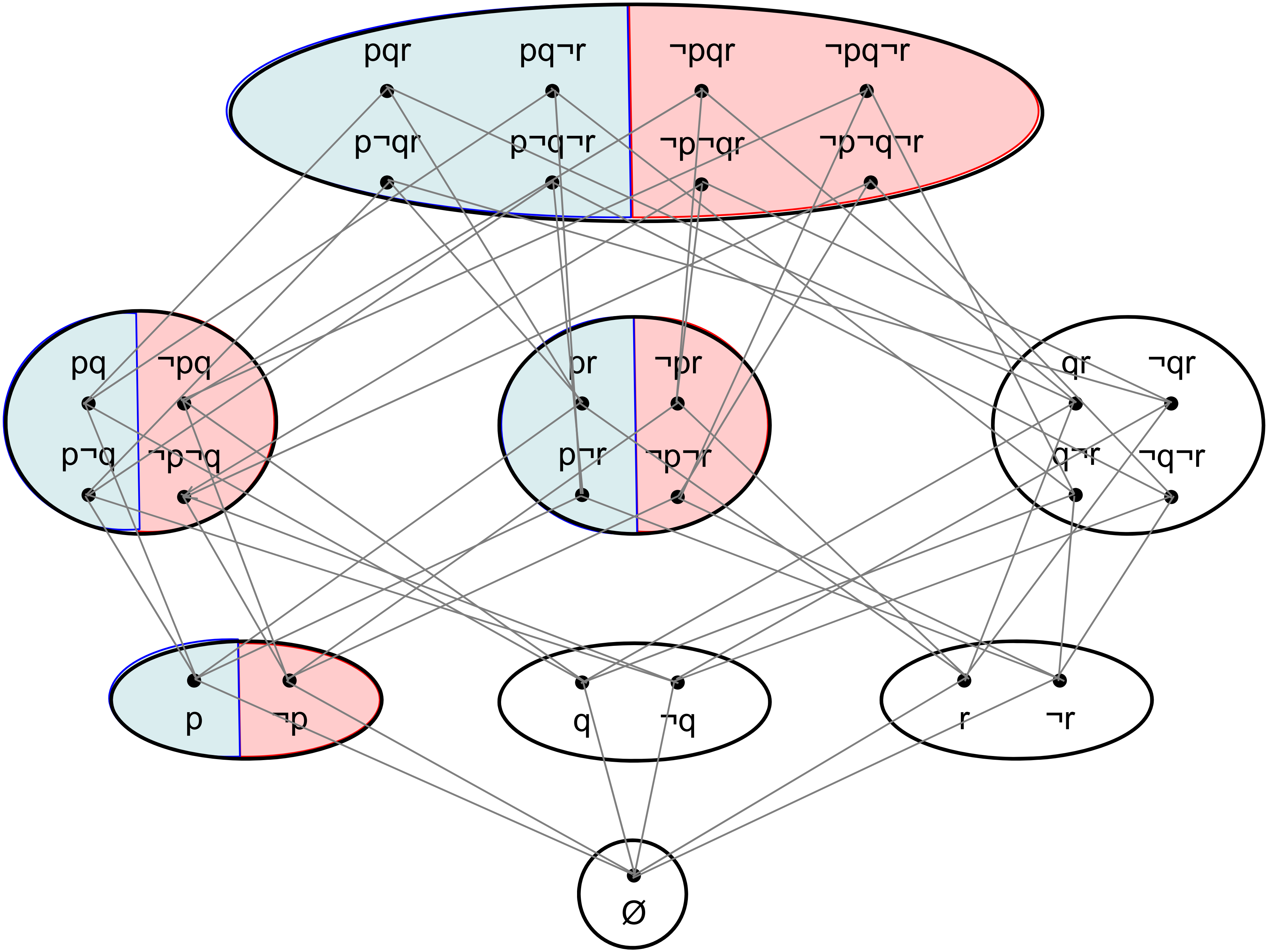
For $D \subseteq S$, $D^{\uparrow} := \bigcup_{S' \in \mathcal{S} : S' \succeq S} \left(r_S^{S'} \right)^{-1} (D)$.

$E \subseteq \Omega$ is an *event* if $E = D^{\uparrow}$ for some *base* $D \subseteq S$
in some *base space* $S \in \mathcal{S}$. $(S(E))$

We write $\emptyset^S$ for the vacuous event with base space S.

Not every subset of Ω is an event.

Negation: For $D^{\uparrow}$ an event,
$\neg D^{\uparrow} := \bigcup_{S' \in \mathcal{S} : S' \succeq S} \left(r_S^{S'} \right)^{-1} (S \setminus D)$.


pqr
pq¬r
¬pqr
¬pq¬r
p¬qr
p¬q¬r
¬p¬qr
¬p¬q¬r
pq
¬pq
p¬q
¬p¬q
pr
¬pr
p¬r
¬p¬r
qr
¬qr
q¬r
¬q¬r
p
¬p
q
¬q
r
¬r
Ø

For $D \subseteq S$, $D^{\uparrow} := \bigcup_{S' \in \mathcal{S} : S' \succeq S} \left(r_S^{S'}\right)^{-1}(D)$.

$E \subseteq \Omega$ is an *event* if $E = D^{\uparrow}$ for some *base* $D \subseteq S$
in some *base space* $S \in \mathcal{S}$. $(S(E))$

We write $\emptyset^S$ for the vacuous event with base space S.

Not every subset of Ω is an event.

Negation: For $D^{\uparrow}$ an event,
$\neg D^{\uparrow} := \bigcup_{S' \in \mathcal{S} : S' \succeq S} \left(r_S^{S'}\right)^{-1}(S \setminus D)$.

For $D \subseteq S$, $D^{\uparrow} := \bigcup_{S' \in \mathcal{S} : S' \succeq S} \left(r_S^{S'} \right)^{-1} (D)$.

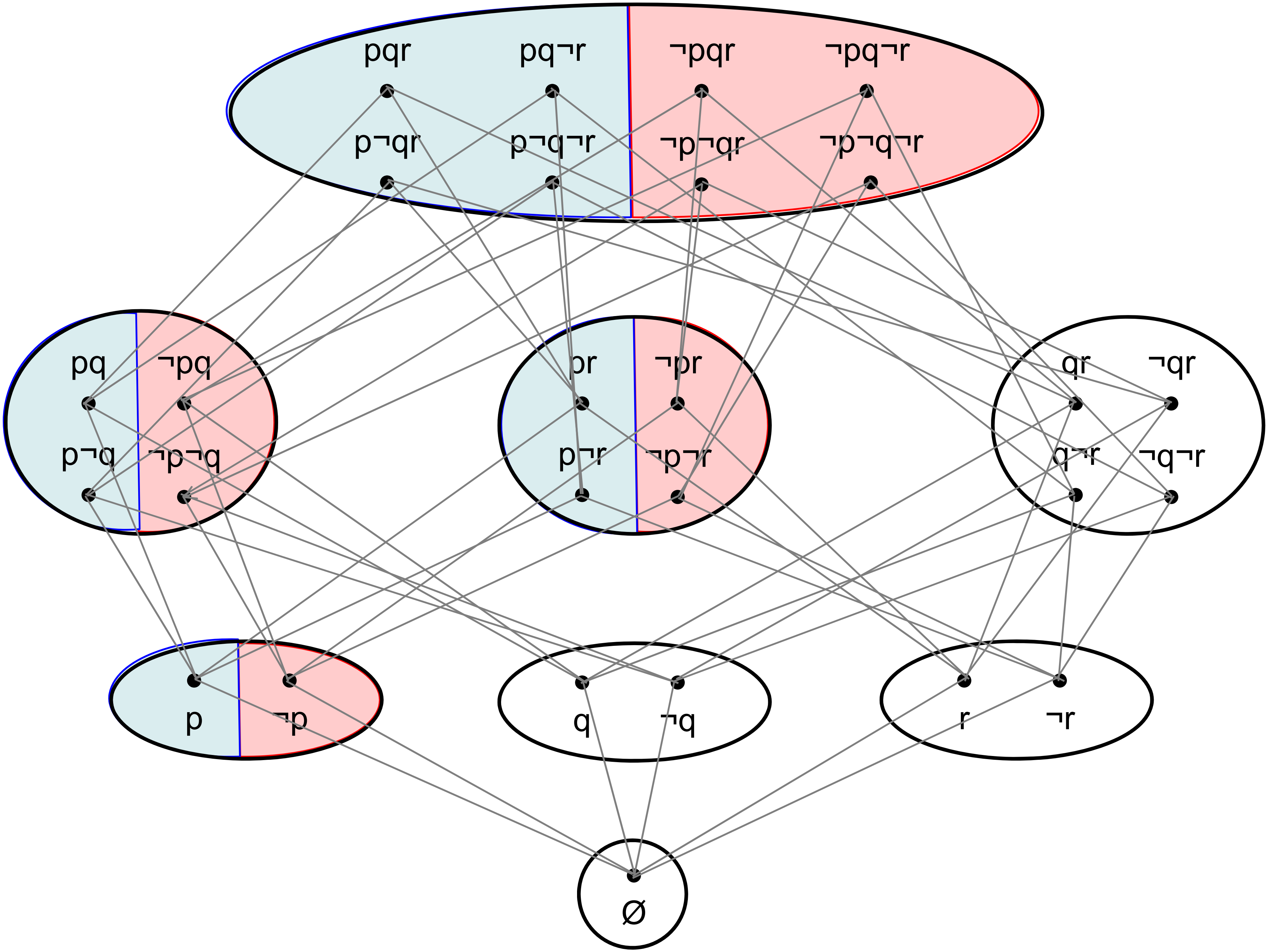
$E \subseteq \Omega$ is an *event* if $E = D^{\uparrow}$ for some *base* $D \subseteq S$
in some *base space* $S \in \mathcal{S}$. $(S(E))$

We write $\emptyset^S$ for the vacuous event with base space S.

Not every subset of Ω is an event.

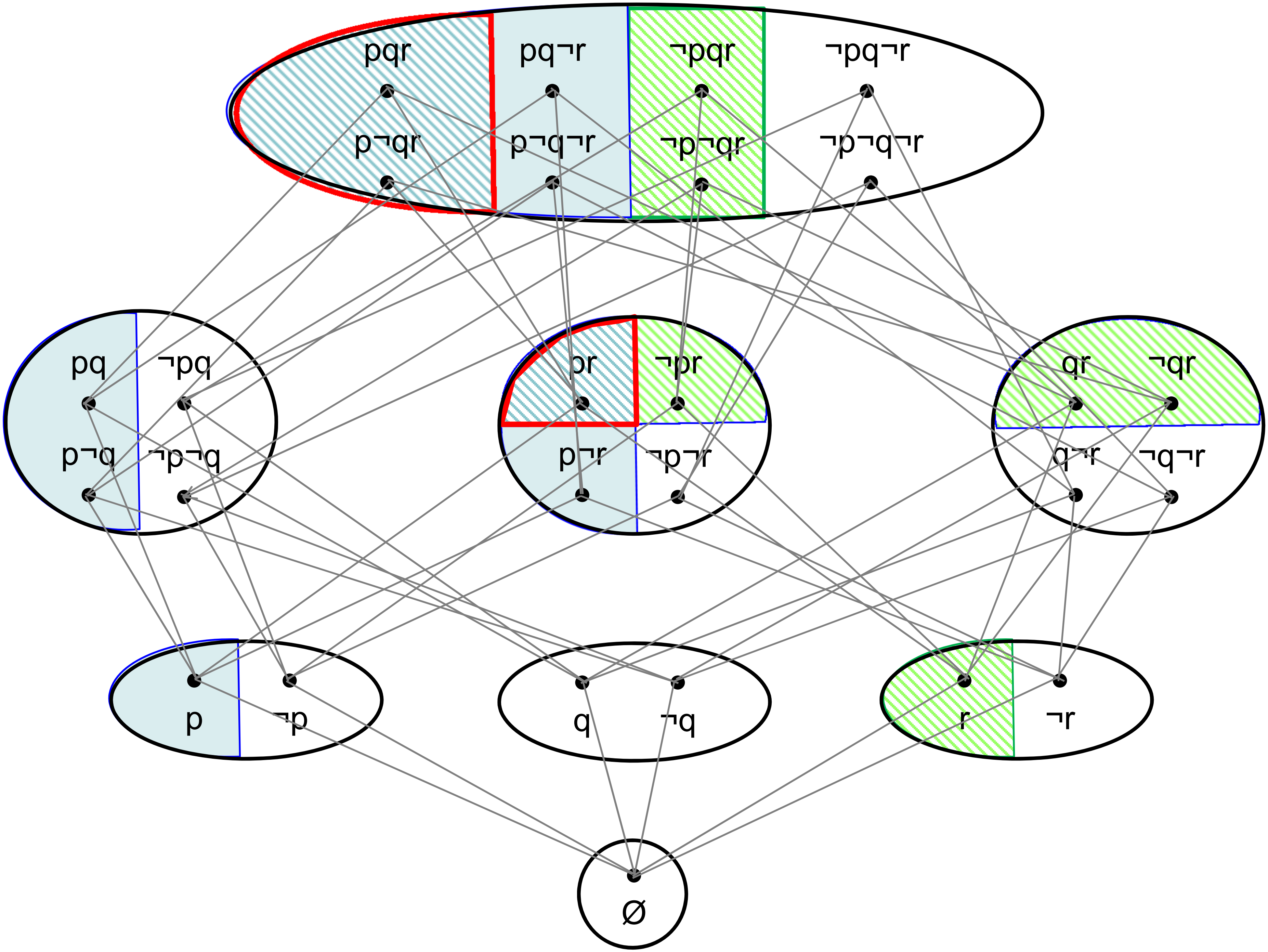
Negation: For $D^{\uparrow}$ an event,
$\neg D^{\uparrow} := \bigcup_{S' \in \mathcal{S} : S' \succeq S} \left(r_S^{S'} \right)^{-1} (S \setminus D)$.

Typcially $\neg E \subsetneqq \Omega \setminus E$.


pqr
pq¬r
¬pqr
¬pq¬r
p¬qr
p¬q¬r
¬p¬qr
¬p¬q¬r
pq
¬pq
p¬q
¬p¬q
pr
¬pr
p¬r
¬p¬r
qr
¬qr
q¬r
¬q¬r
p
¬p
q
¬q
r
¬r
Ø

$\{E_j\}$ collection of events

Conjunction: $\bigwedge_j E_j := \bigcap_j E_j$

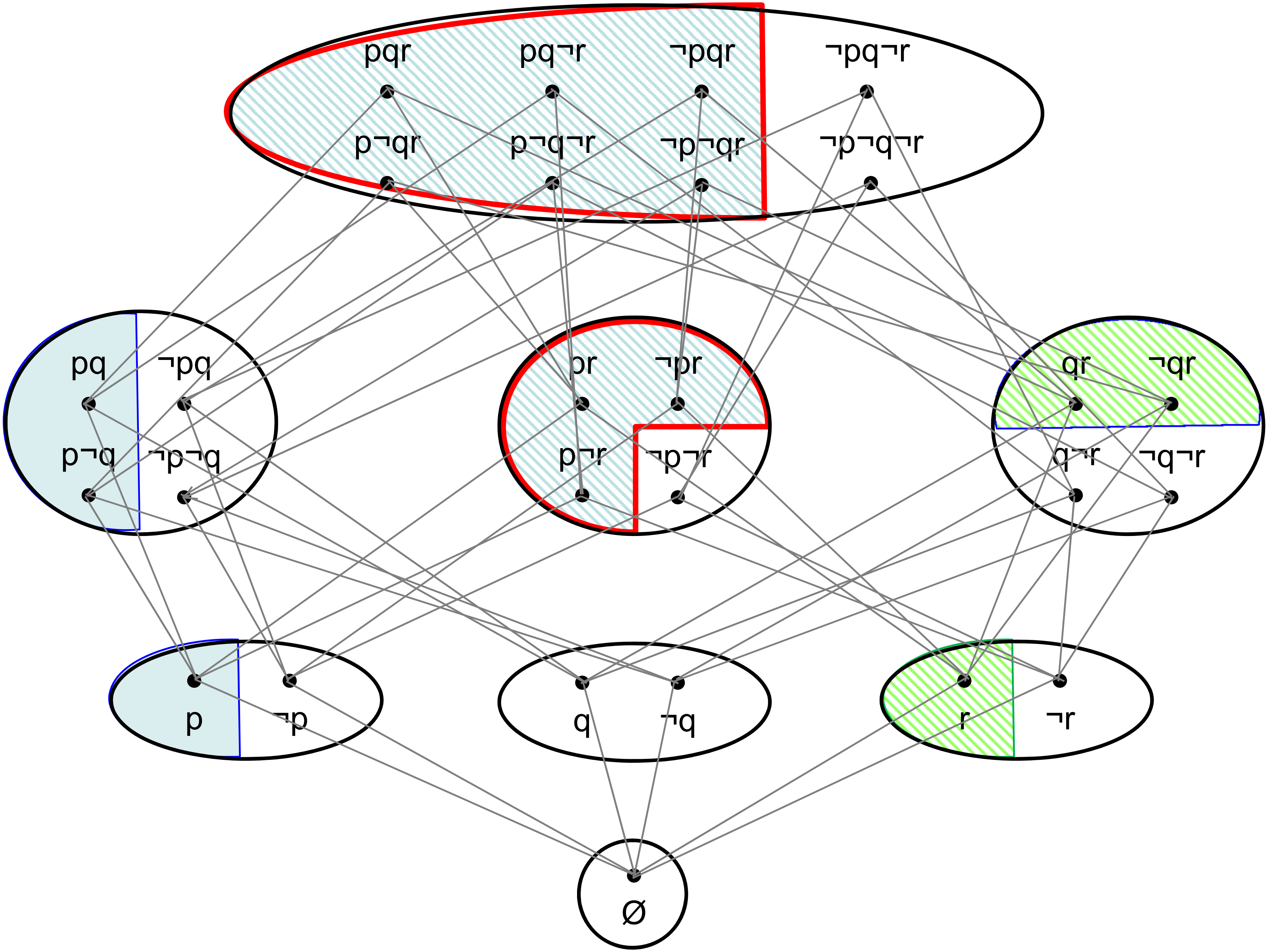

pqr
pq¬r
¬pqr
¬pq¬r
p¬qr
p¬q¬r
¬p¬qr
¬p¬q¬r
pq
¬pq
p¬q
¬p¬q
pr
¬pr
p¬r
¬p¬r
qr
¬qr
q¬r
¬q¬r
p
¬p
q
¬q
r
¬r
Ø

$\{E_j\}$ collection of events

Conjunction: $\bigwedge_j E_j := \bigcap_j E_j$

Disjunction: $\bigvee_j E_j := \neg \left(\bigwedge_j \neg E_j \right)$

Typically $\bigvee_j E_j \subsetneqq \bigcup_j E_j$


pqr
pq¬r
¬pqr
¬pq¬r
p¬qr
p¬q¬r
¬p¬qr
¬p¬q¬r
pq
¬pq
p¬q
¬p¬q
pr
¬pr
p¬r
¬p¬r
qr
¬qr
q¬r
¬q¬r
p
¬p
q
¬q
r
¬r
Ø

$\{E_j\}$ collection of events

Conjunction: $\bigwedge_j E_j := \bigcap_j E_j$

Disjunction: $\bigvee_j E_j := \neg\left(\bigwedge_j \neg E_j\right)$

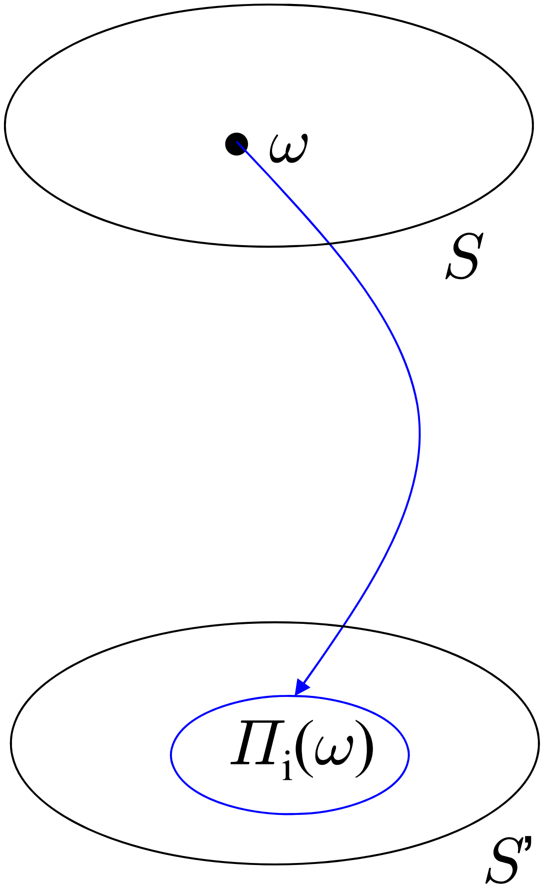
Typically $\bigvee_j E_j \subsetneqq \bigcup_j E_j$

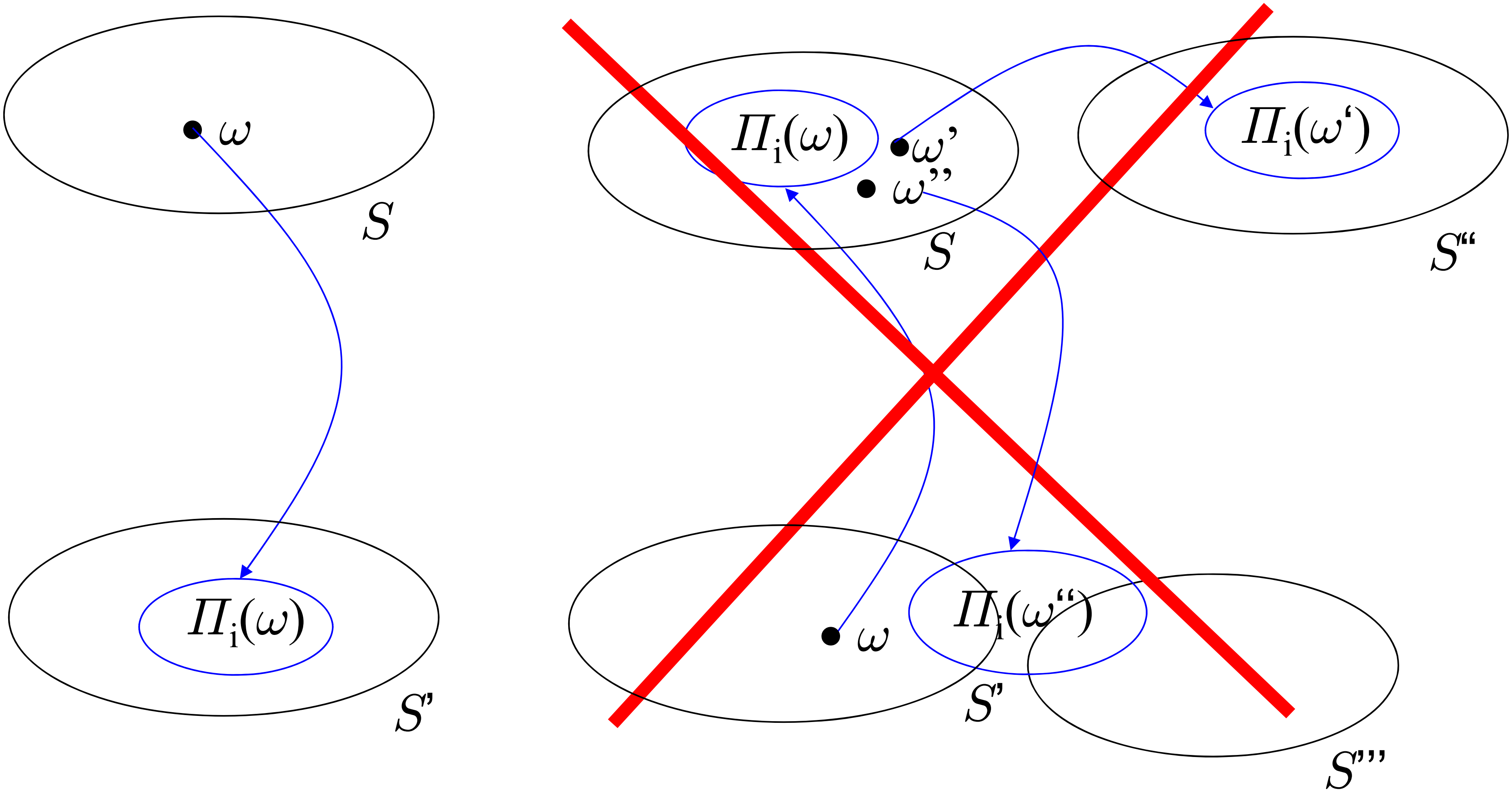
$E \subseteq F$ if and olny if $E \subseteq F$ and $S(E) \succeq S(F)$.

Set of all events is not necessarily an algebra
(many vacuous events).

For each $i \in I$, there is a possibility correspondence $\Pi_i : \Omega \longrightarrow 2^\Omega \setminus \{\emptyset\}$ satisfying the following properties:

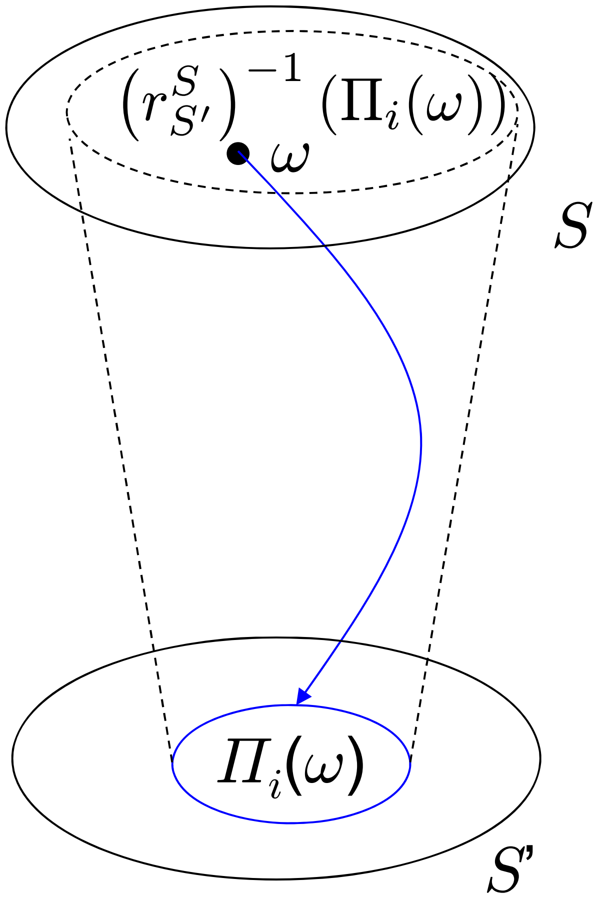
0. Confinedment: If $\omega \in S$ then $\Pi_i(\omega) \subseteq S'$ for some $S' \preceq S$.

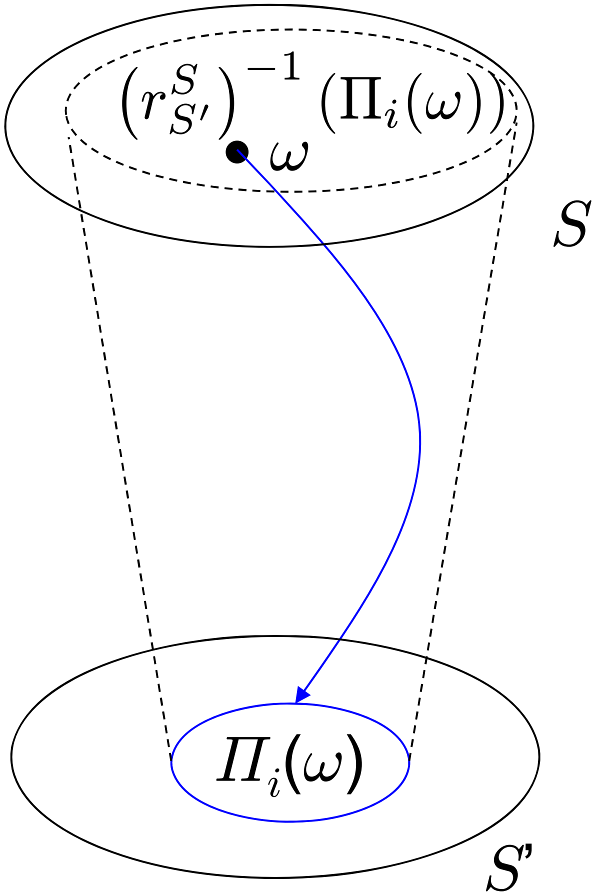

ω
S
$\Pi_i(\omega)$
S'

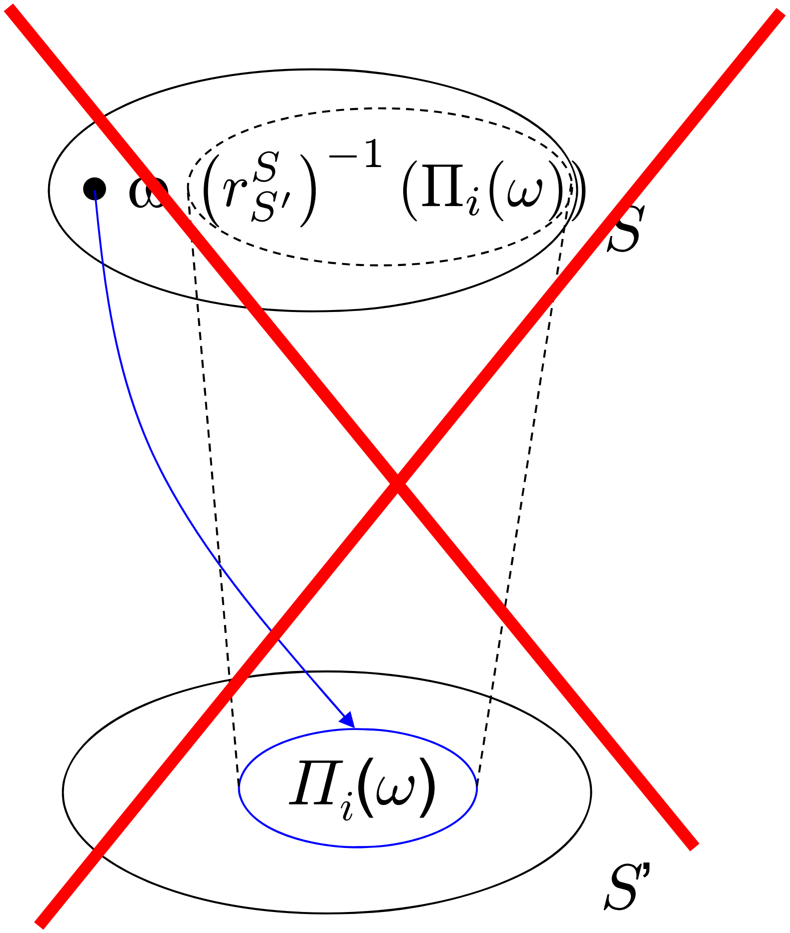

ω
S
Π_i(ω)
S'
Π_i(ω)
ω'
ω''
S
Π_i(ω')
S"
ω
Π_i(ω")
S'
S'''

For each $i \in I$, there is a possibility correspondence $\Pi_i : \Omega \longrightarrow 2^{\Omega} \setminus \{\emptyset\}$ satisfying the following properties:

0. Confinedment: If $\omega \in S$ then $\Pi_i(\omega) \subseteq S'$ for some $S' \preceq S$.
1. Generalized Reflexivity: $\omega \in \Pi_i^{\uparrow}(\omega)$ for every $\omega \in \Omega$.
2. Stationarity: $\omega' \in \Pi_i(\omega)$ implies $\Pi_i(\omega') = \Pi_i(\omega)$.
3. Projections Preserve Awareness: If $\omega \in S'$, $\omega \in \Pi_i(\omega)$ and $S \preceq S'$ then $\omega_S \in \Pi_i(\omega_S)$.
4. Projections Preserve Ignorance: If $\omega \in S'$ and $S \preceq S'$ then $\Pi_i^{\uparrow}(\omega) \subseteq \Pi_i^{\uparrow}(\omega_S)$.
5. Projections Preserve Knowledge: If $S \preceq S' \preceq S''$, $\omega \in S''$ and $\Pi_i(\omega) \subseteq S'$ then $\left(\Pi_i(\omega)\right)_S = \Pi_i(\omega_S)$.


$\left(r_{S'}^{S}\right)^{-1}(\Pi_i(\omega))$
ω
S
$\Pi_i(\omega)$
S'


$\left(r^{S}_{S'}\right)^{-1}(\Pi_i(\omega))$
ω
S
$\Pi_i(\omega)$
S'

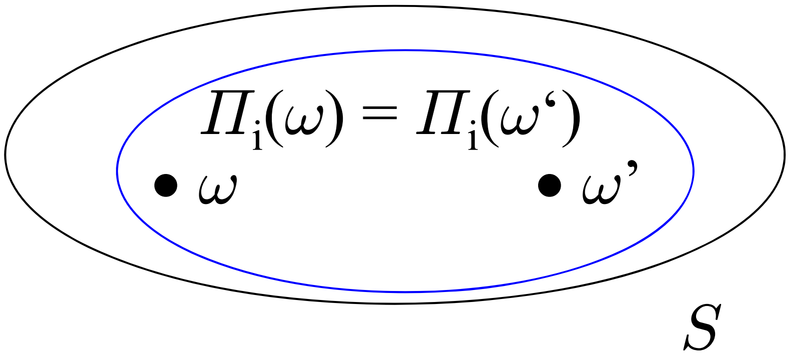

ω
$\left(r^{S}_{S'}\right)^{-1}(\Pi_i(\omega))$
S
$\Pi_i(\omega)$
S'

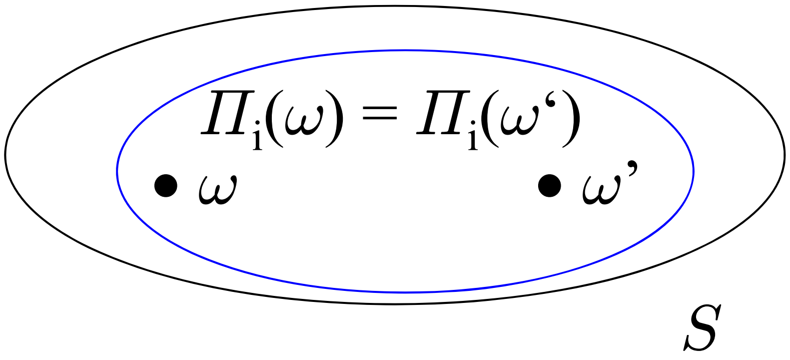
For each $i \in I$, there is a possibility correspondence $\Pi_i : \Omega \longrightarrow 2^{\Omega} \setminus \{\emptyset\}$ satisfying the following properties:

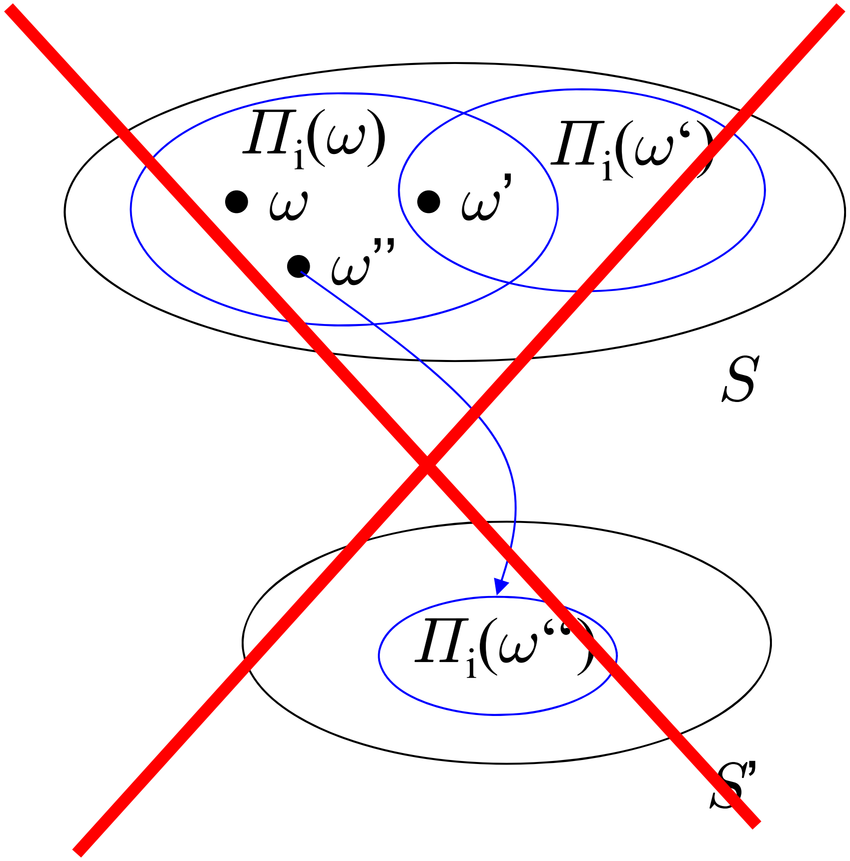
0. Confinedment: If $\omega \in S$ then $\Pi_i(\omega) \subseteq S'$ for some $S' \preceq S$.

1. Generalized Reflexivity: $\omega \in \Pi_i^{\uparrow}(\omega)$ for every $\omega \in \Omega$.

2. Stationarity: $\omega' \in \Pi_i(\omega)$ implies $\Pi_i(\omega') = \Pi_i(\omega)$.

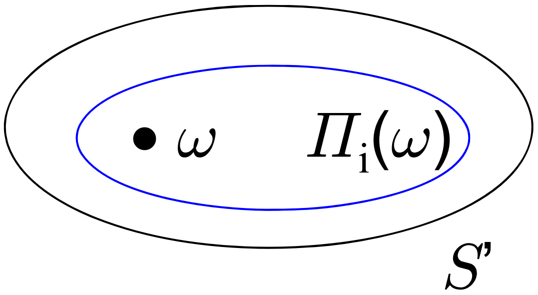

$\Pi_i(\omega) = \Pi_i(\omega')$
ω
ω'
S

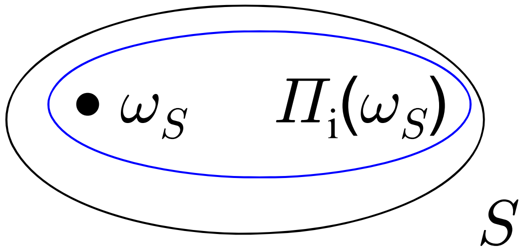

Πi(ω) = Πi(ω‘)
ω
ω’
S

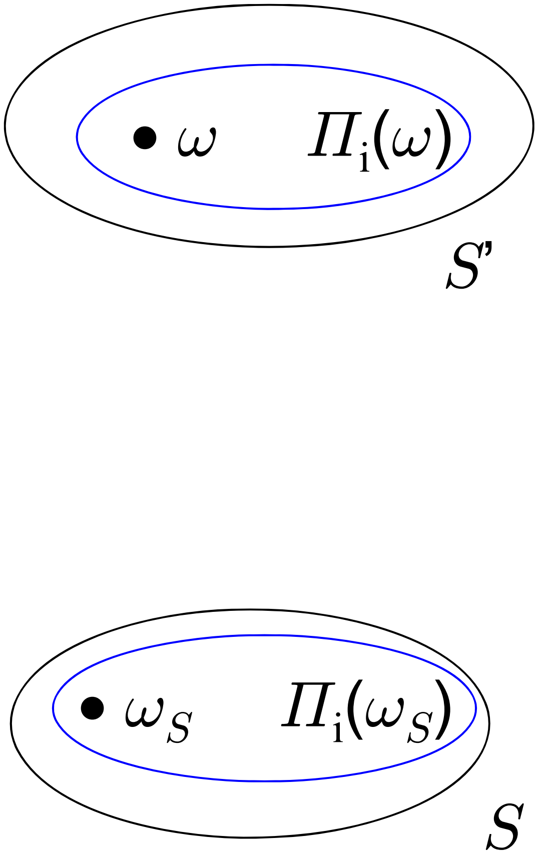

Πi(ω)
ω
ω’
ω’’
Πi(ω‘)
S
Πi(ω‘‘)
S’

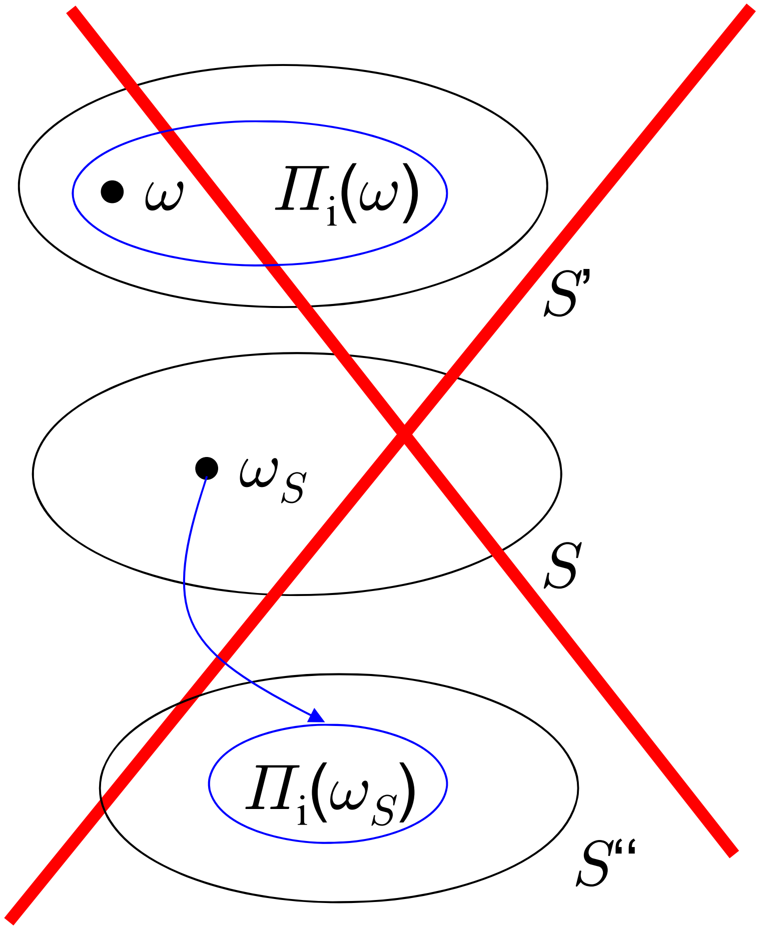
For each $i \in I$, there is a possibility correspondence $\Pi_i : \Omega \longrightarrow 2^{\Omega} \setminus \{\emptyset\}$ satisfying the following properties:

0. Confinedment: If $\omega \in S$ then $\Pi_i(\omega) \subseteq S'$ for some $S' \preceq S$.
1. Generalized Reflexivity: $\omega \in \Pi_i^{\uparrow}(\omega)$ for every $\omega \in \Omega$.
2. Stationarity: $\omega' \in \Pi_i(\omega)$ implies $\Pi_i(\omega') = \Pi_i(\omega)$.
3. Projections Preserve Awareness: If $\omega \in S'$, $\omega \in \Pi_i(\omega)$ and $S \preceq S'$ then $\omega_S \in \Pi_i(\omega_S)$.


ω
$\Pi_{\mathrm{i}}(\omega)$
S'

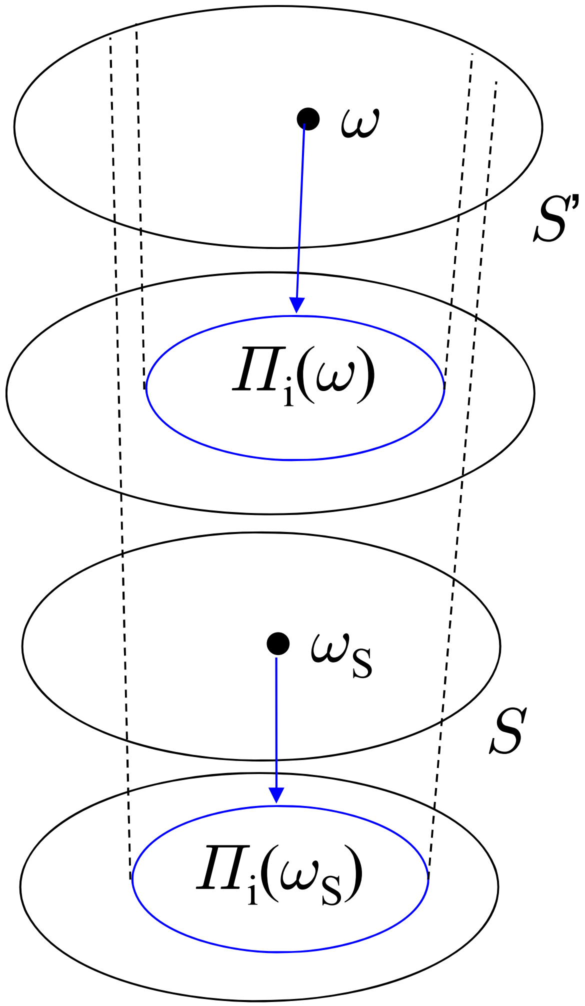

ω_S
$\Pi_{\mathrm{i}}(\omega_S)$
S


• ω
Π_i(ω)
S'
• ω_S
Π_i(ω_S)
S

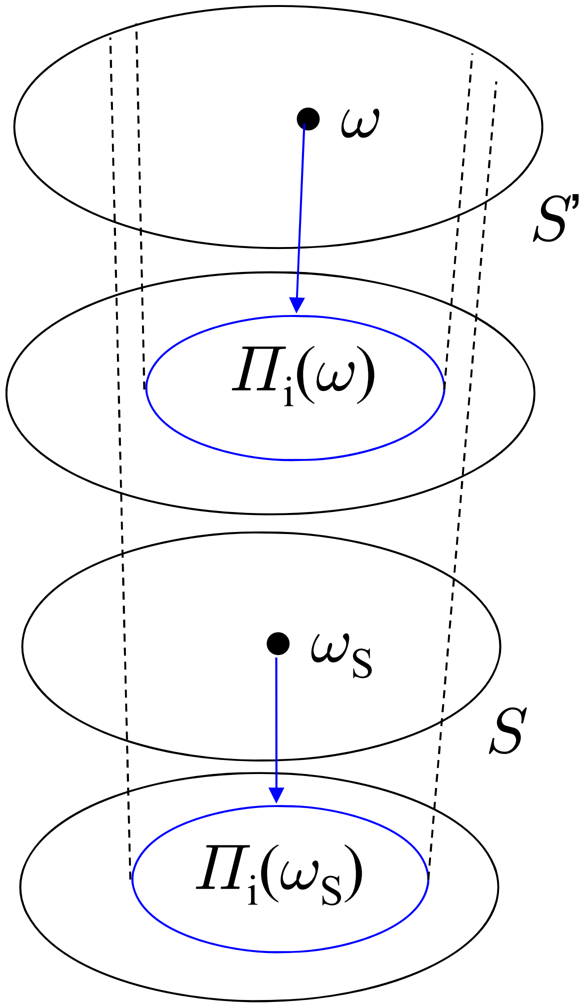

• ω
Π_i(ω)
S'
• ω_S
S
Π_i(ω_S)
S''

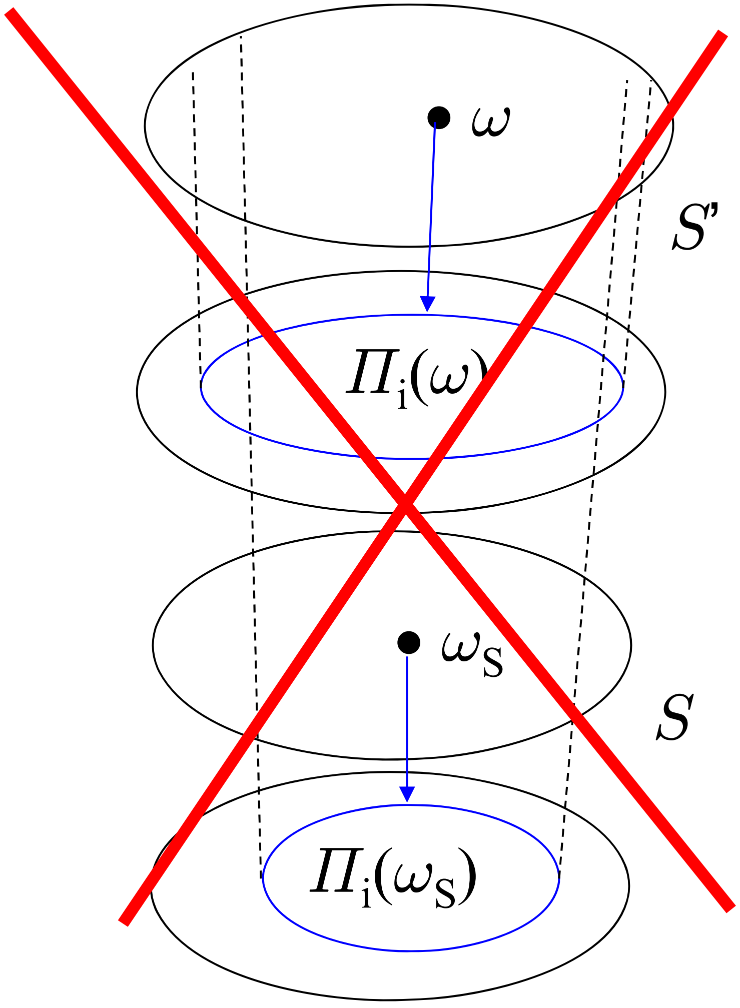
For each $i \in I$, there is a possibility correspondence $\Pi_i : \Omega \longrightarrow 2^\Omega \setminus \{\emptyset\}$ satisfying the following properties:

0. Confinedment: If $\omega \in S$ then $\Pi_i(\omega) \subseteq S'$ for some $S' \preceq S$.
1. Generalized Reflexivity: $\omega \in \Pi_i^{\uparrow}(\omega)$ for every $\omega \in \Omega$.
2. Stationarity: $\omega' \in \Pi_i(\omega)$ implies $\Pi_i(\omega') = \Pi_i(\omega)$.
3. Projections Preserve Awareness: If $\omega \in S'$, $\omega \in \Pi_i(\omega)$ and $S \preceq S'$ then $\omega_S \in \Pi_i(\omega_S)$.
4. Projections Preserve Ignorance: If $\omega \in S'$ and $S \preceq S'$ then $\Pi_i^{\uparrow}(\omega) \subseteq \Pi_i^{\uparrow}(\omega_S)$.



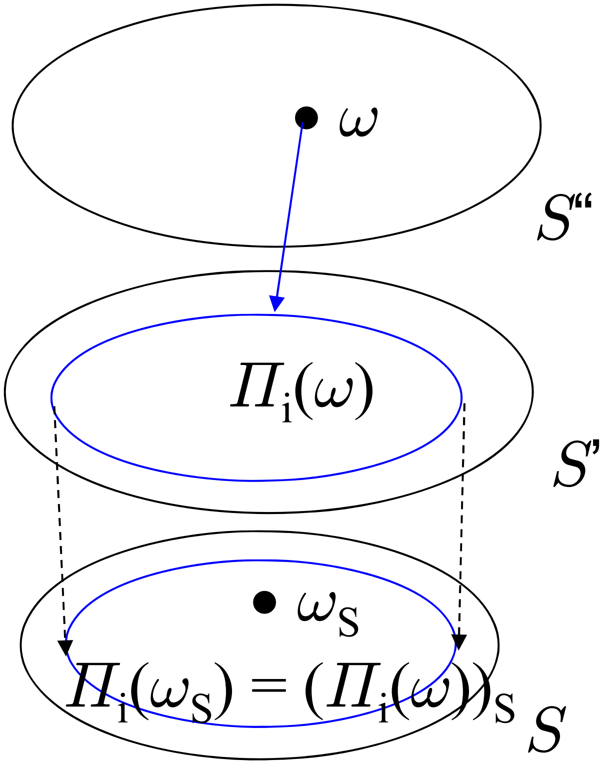
ω
S'
$\Pi_i(\omega)$
ω_S
S
$\Pi_i(\omega_S)$


ω
S'
$\Pi_i(\omega)$
ω_S
S
$\Pi_i(\omega_S)$

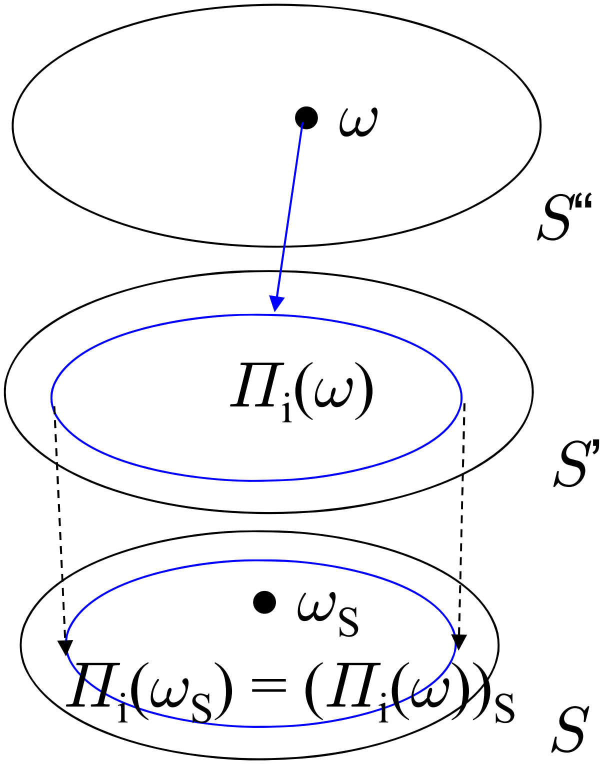

ω
S'
$\Pi_i(\omega)$
ω_S
S
$\Pi_i(\omega_S)$

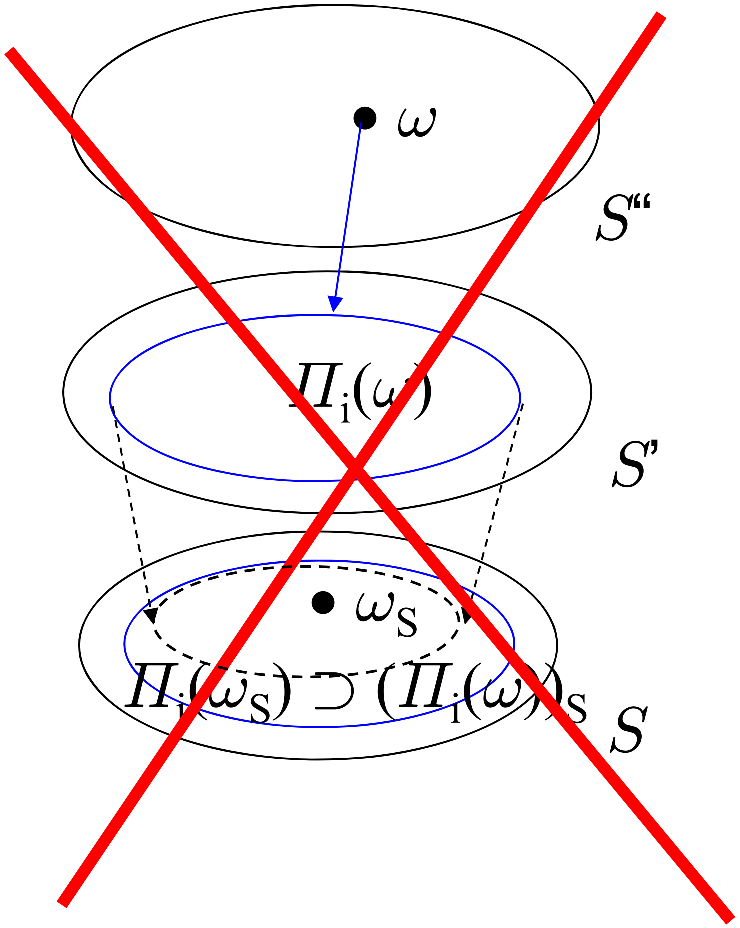
For each $i \in I$, there is a possibility correspondence $\Pi_i : \Omega \longrightarrow 2^{\Omega} \setminus \{\emptyset\}$ satisfying the following properties:

0. Confinedment: If $\omega \in S$ then $\Pi_i(\omega) \subseteq S'$ for some $S' \preceq S$.
1. Generalized Reflexivity: $\omega \in \Pi_i^{\uparrow}(\omega)$ for every $\omega \in \Omega$.
2. Stationarity: $\omega' \in \Pi_i(\omega)$ implies $\Pi_i(\omega') = \Pi_i(\omega)$.
3. Projections Preserve Awareness: If $\omega \in S'$, $\omega \in \Pi_i(\omega)$ and $S \preceq S'$ then $\omega_S \in \Pi_i(\omega_S)$.
4. Projections Preserve Ignorance: If $\omega \in S'$ and $S \preceq S'$ then $\Pi_i^{\uparrow}(\omega) \subseteq \Pi_i^{\uparrow}(\omega_S)$.
5. Projections Preserve Knowledge: If $S \preceq S' \preceq S''$, $\omega \in S''$ and $\Pi_i(\omega) \subseteq S'$ then $\left(\Pi_i(\omega)\right)_S = \Pi_i(\omega_S)$.



ω
S''
$\Pi_i(\omega)$
S'
ω_S
$\Pi_i(\omega_S) = (\Pi_i(\omega))_S$
S


ω
S“
$\Pi_i(\omega)$
S’
ω_S
$\Pi_i(\omega_S) = (\Pi_i(\omega))_S$
S


ω
S“
$\Pi_i(\omega)$
S’
ω_S
$\Pi_i(\omega_S) \supset (\Pi_i(\omega))_S$
S

For each $i \in I$, there is a possibility correspondence $\Pi_i : \Omega \longrightarrow 2^{\Omega} \setminus \{\emptyset\}$ satisfying the following properties:

0. Confinedment: If $\omega \in S$ then $\Pi_i(\omega) \subseteq S'$ for some $S' \preceq S$.
1. Generalized Reflexivity: $\omega \in \Pi_i^{\uparrow}(\omega)$ for every $\omega \in \Omega$.
2. Stationarity: $\omega' \in \Pi_i(\omega)$ implies $\Pi_i(\omega') = \Pi_i(\omega)$.
3. Projections Preserve Awareness: If $\omega \in S'$, $\omega \in \Pi_i(\omega)$ and $S \preceq S'$ then $\omega_S \in \Pi_i(\omega_S)$.
4. Projections Preserve Ignorance: If $\omega \in S'$ and $S \preceq S'$ then $\Pi_i^{\uparrow}(\omega) \subseteq \Pi_i^{\uparrow}(\omega_S)$.
5. Projections Preserve Knowledge: If $S \preceq S' \preceq S''$, $\omega \in S''$ and $\Pi_i(\omega) \subseteq S'$ then $\left(\Pi_i(\omega)\right)_S = \Pi_i(\omega_S)$.

Remark Generalized Reflexivity implies that if $S' \preceq S$, $\omega \in S$ and $\Pi_i(\omega) \subseteq S'$, then $r^S_{S'}(\omega) \in \Pi_i(\omega)$. In particular, we have $\Pi_i(\omega) \neq \emptyset$, for all $\omega \in \Omega$.

Remark Property 5 and Confinement (Property 0) imply Property 3.

Remark Generalized Reflexivity implies that if $S' \preceq S$, $\omega \in S$ and $\Pi_i(\omega) \subseteq S'$, then $r^{S}_{S'}(\omega) \in \Pi_i(\omega)$. In particular, we have $\Pi_i(\omega) \neq \emptyset$, for all $\omega \in \Omega$.

Remark Property 5 and Confinement (Property 0) imply Property 3.

$\left\langle \langle \mathcal{S}, \preceq \rangle, (r^{S'}_{S})_{S \preceq S'}, (\Pi_i)_{i \in I} \right\rangle$ is an *unawareness structure.*

The *knowledge operator* of individual i on events E is defined, as usual, by

$$K_i(E) := \{\omega \in \Omega : \Pi_i(\omega) \subseteq E\}\,,$$

if there is a state ω such that $\Pi_i(\omega) \subseteq E$, and by

$$K_i(E) := \emptyset^{S(E)}$$

otherwise.

The *knowledge operator* of individual i on events E is defined, as usual, by

$$K_i(E) := \{\omega \in \Omega : \Pi_i(\omega) \subseteq E\}\,,$$

if there is a state ω such that $\Pi_i(\omega) \subseteq E$, and by

$$K_i(E) := \emptyset^{S(E)}$$

otherwise.

Proposition If E is an event, then $K_i(E)$ is an $S(E)$-based event.

Proposition The knowledge operator K_i has the following properties:

(i) Necessitation: $K_i(\Omega) = \Omega$,

(ii) Conjunction: $K_i\left(\bigcap_j E_j\right) = \bigcap_j K_i\left(E_j\right)$,

(iii) Truth: $K_i(E) \subseteq E$,

(iv) Positive Introspection: $K_i(E) \subseteq K_i K_i(E)$,

(v) Monotonicity: $E \subseteq F$ implies $K_i(E) \subseteq K_i(F)$,

(vi) $\neg K_i(E) \cap \neg K_i \neg K_i(E) \subseteq \neg K_i \neg K_i \neg K_i(E)$.

The *awareness operator* of individual i is defined on events E by

$$A_i(E) = \{\omega \in \Omega : \Pi_i(\omega) \subseteq S(E)\},$$

if there is a state $\omega \in \Omega$ such that $\Pi_i(\omega) \subseteq S(E)$, and by

$$A_i(E) = \emptyset^{S(E)}$$

otherwise.

The *awareness operator* of individual i is defined on events E by

$$A_i(E) = \left\{\omega \in \Omega : \Pi_i(\omega) \subseteq S(E)^{\uparrow}\right\},$$

if there is a state $\omega \in \Omega$ such that $\Pi_i(\omega) \subseteq S(E)$, and by

$$A_i(E) = \emptyset^{S(E)}$$

otherwise.

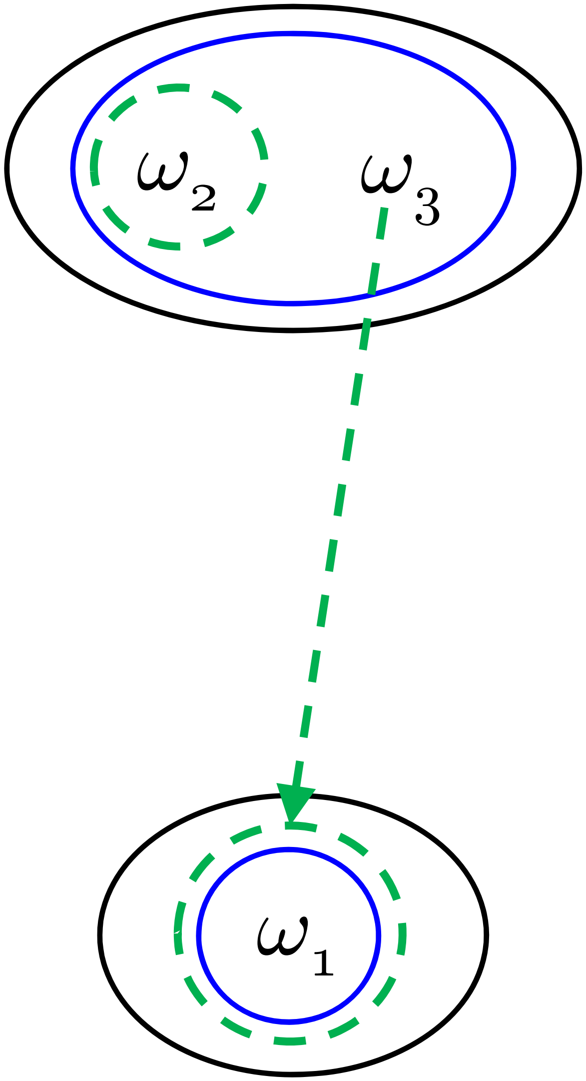
The *unawareness operator* is then naturally defined by

$$U_i(E) = \neg A_i(E).$$

Proposition The following properties obtain:

1. KU Introspection: $K_iU_i(E) = \emptyset^{S(E)}$,
2. AU Introspection: $U_i(E) = U_iU_i(E)$,
3. Plausibility: $U_i(E) = \neg K_i(E) \cap \neg K_i \neg K_i(E)$,
4. Strong Plausibility: $U_i(E) = \bigcap_{n=1}^{\infty} \left(\neg K_i\right)^n (E)$,
5. Weak Necessitation: $A_i(E) = K_i \left(S\left(E\right)^{\uparrow}\right)$,
6. Weak Negative Introspection: $\neg K_i(E) \cap A_i \neg K_i(E) = K_i \neg K_i(E)$,
7. Symmetry: $A_i(E) = A_i(\neg E)$,
8. A-Conjunction: $\bigcap_{\lambda \in L} A_i\left(E_\lambda\right) = A_i\left(\bigcap_{\lambda \in L} E_\lambda\right)$,
9. AK- Reflection: $A_i(E) = A_iK_i(E)$,
10. AA- Reflection: $A_i(E) = A_iA_i(E)$,
11. A-Introspection: $A_i(E) = K_iA_i(E)$.

Illustration



$E = \{\omega_2\}$

$A_b(E)=\{\omega_2\}$
$U_b(E) = \{\omega_3\}$
$\neg K_a U_b(E) = \{\omega_2, \omega_3\}$
$\neg K_a A_b(E) = \{\omega_2, \omega_3\}$
$K_b \neg K_a U_b(E) = \{\omega_2\}$
$A_a K_b \neg K_a U_b(E) = \{\omega_2, \omega_3\}$
…

The mutual knowledge operator on events is defined by

$$K(E) = \bigcap_{i \in I} K_i(E).$$

The mutual knowledge operator on events is defined by

$$K(E) = \bigcap_{i \in I} K_i(E).$$

The common knowledge operator is defined by

$$CK(E) = \bigcap_{n=1}^{\infty} K^n(E).$$

The mutual knowledge operator on events is defined by

$$K(E) = \bigcap_{i \in I} K_i(E).$$

The common knowledge operator is defined by

$$CK(E) = \bigcap_{n=1}^{\infty} K^n(E).$$

The mutual awareness operator on events is defined by

$$A(E) = \bigcap_{i \in I} A_i(E),$$

and the common awareness operator by

$$CA(E) = \bigcap_{n=1}^{\infty} (A)^n (E).$$

Proposition The following multi-agent properties obtain:

1. $A_i(E) = A_i A_j(E)$,
2. $A_i(E) = A_i K_j(E)$,
3. $K_i(E) \subseteq A_i K_j(E)$,
4. $A(E) = K(S(E)^{\uparrow})$,
5. $A(E) = CA(E)$,
6. $K(E) \subseteq A(E)$,
7. $CK(E) \subseteq CA(E)$,
8. $CK(S(E)^{\uparrow}) \subseteq CA(E)$.

Logic of unawareness

- Syntax versus semantics: What’s the internal structure of states? What’s the interpretation of the lattice?
- How comprehensive are unawareness structures? Can the model itself be subject to agents’ uncertainties? Can any minute detail of awareness, beliefs, mutual beliefs, etc. of all individuals be described in some state of the model?
- Soundness: Is any theorem valid in all states?
- Completeness: Is every formula that is valid in all states also provable?

Given a nonempty set of agents $i \in I$, a nonempty set of atomic formulae $p \in \mathrm{At}$ as well as the special formula $\top$, the formulae φ of the language $L_I^{k,a}(\mathrm{At})$ are defined by the grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \psi \mid k_i\varphi \mid a_i\varphi.$$

atomic formula: “penicillium rubens has antibiotic properties”

Given a nonempty set of agents $i \in I$, a nonempty set of atomic formulae $p \in \mathrm{At}$ as well as the special formula $\top$, the formulae φ of the language $L_I^{k,a}(\mathrm{At})$ are defined by the grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \psi \mid k_i\varphi \mid a_i\varphi.$$

atomic formula: “penicillium rubens has antibiotic properties”

$\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$
$(\varphi \Rightarrow \psi) := (\neg\varphi \vee \psi)$
$(\varphi \Leftrightarrow \psi) := (\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)$

Define inductively the set of primitive propositions $\mathrm{At}(\varphi)$ that appear in φ

- $\mathrm{At}(\top) := \emptyset$,
- $\mathrm{At}(p) := p$, for $p \in \mathrm{At}$,
- $\mathrm{At}(\neg\varphi) := \mathrm{At}(\varphi)$,
- $\mathrm{At}(\varphi \wedge \psi) := \mathrm{At}(\varphi) \cup \mathrm{At}(\psi)$,
- $\mathrm{At}(k_i\varphi) := \mathrm{At}(\varphi) =: \mathrm{At}(a_i\varphi)$.

Define inductively the set of primitive propositions $\mathrm{At}(\varphi)$ that appear in φ

- $\mathrm{At}(\top) := \emptyset$,
- $\mathrm{At}(p) := p$, for $p \in \mathrm{At}$,
- $\mathrm{At}(\neg\varphi) := \mathrm{At}(\varphi)$,
- $\mathrm{At}(\varphi \wedge \psi) := \mathrm{At}(\varphi) \cup \mathrm{At}(\psi)$,
- $\mathrm{At}(k_i\varphi) := \mathrm{At}(\varphi) =: \mathrm{At}(a_i\varphi)$.

Modica-Rustichini definition of awareness: $a_i\varphi := k_i\varphi \vee k_i\neg k_i\varphi$.

An *axiom* is a formula assumed.

An *inference rule* infers a formula (i.e., a conclusion) from a collection of formulae (i.e., the hypothesis).

An *axiom system* consists of a collection of axioms and inferences rules.

Axiom system $S5_I^{k,a}$:

Prop. All substitutions instances of tautologies of propositional logic, including the formula $\top$.

AS. $a_i \neg \varphi \Leftrightarrow a_i \varphi$ (Symmetry)

AC. $a_i(\varphi \wedge \psi) \Leftrightarrow a_i \varphi \wedge a_i \psi$ (Awareness Conjunction)

A$_i$K$_j$R. $a_i \varphi \Leftrightarrow a_i k_j \varphi$, for all $j \in I$ (Awareness Knowledge Reflection)

T. $k_i \varphi \Rightarrow \varphi$ (Axiom of Truth)

4. $k_i \varphi \Rightarrow k_i k_i \varphi$ (Positive Introspection Axiom)

MP. From φ and $\varphi \Rightarrow \psi$ infer ψ (modus ponens)

RK. For all natural numbers $n \geq 1$, if $\varphi_1, \varphi_2, ..., \varphi_n$ and φ are such that $\mathrm{At}(\varphi) \subseteq \bigcup_{\ell=1}^n \mathrm{At}(\varphi_\ell)$, then $\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n \Rightarrow \varphi$ implies $k_i \varphi_1 \wedge k_i \varphi_2 \wedge \cdots \wedge k_i \varphi_n \Rightarrow k_i \varphi$. (RK-Inference)

A *proof* in an axiom system consists of a sequence of formulae, where each formula is either an axiom in the axiom system or follows by an application of an inference rule.

A proof is a proof of a formula φ if the last formula in the proof is φ.

A formula φ is *provable* in an axiom system if there is a proof of φ in the axiom system.

The set of *theorems* of an axiom system is the smallest set of formulae that contain all axioms and that is closed under inference rules of the axiom system.

Define $u_i\varphi := \neg a_i\varphi$.

Remark The Modica and Rustichini definition of awareness and axiom system $S5_I^{k,a}$ implies:

K. $k_i\varphi \wedge k_i(\varphi \Rightarrow \psi) \Rightarrow k_i\varphi$

$k_i\varphi \wedge k_i\psi \Rightarrow k_i(\varphi \wedge \psi)$

NNI. $u_i\varphi \Rightarrow \neg k_i\neg k_i\neg k_i\varphi$

AI. $a_i\varphi \Rightarrow k_i a_i\varphi$

AGPP. $a_i\varphi \Leftrightarrow \bigwedge_{p\in \mathrm{At}(\varphi)} a_i p$

Gen$_A$. If φ is a theorem, then $a_i\varphi \Rightarrow k_i\varphi$ is a theorem.

Given a language $L(\mathrm{At})$, a set of formulae Γ is *consistent* with respect to an axiom system if and only if there is no formula φ such that both φ and $\neg\varphi$ are provable from Γ.

ω is *maximally consistent* of formulae if it is consistent and for any formula $\varphi \in L(\mathrm{At}) \setminus \omega$, the set $\omega \cup \{\varphi\}$ is not consistent.

Every consistent subset of $L(\mathrm{At})$ can be extended to a maximally consistent subset ω of $L(\mathrm{At})$. Moreover, $\Gamma \subseteq L(\mathrm{At})$ is a maximally consistent subset of $L(\mathrm{At})$ if and only if Γ is consistent and for every $\varphi \in \mathsf{L}(\mathsf{At})$, $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

Canonical construction:

For every $\mathrm{At}' \subseteq \mathsf{At}$, let $S_{\mathrm{At}'}$ be the set of maximally consistent sets $\omega_{\mathrm{At}'}$ of formulae in the sublanguage $L_I^{k,a}(\mathrm{At}')$.

Canonical construction:

For every $\mathrm{At}' \subseteq \mathsf{At}$, let $S_{\mathrm{At}'}$ be the set of maximally consistent sets $\omega_{\mathrm{At}'}$ of formulae in the sublanguage $L_I^{k,a}(\mathrm{At}')$.

$\langle \{S_{\mathrm{At}'}\}_{\mathrm{At}' \subseteq \mathsf{At}}, \preceq \rangle$ is a complete lattice of disjoint spaces with the order defined by $S_{\mathrm{At}''} \succeq S_{\mathsf{At}'}$ if and only if $\mathsf{At}'' \supseteq \mathsf{At}'$.

$\Omega := \bigcup_{\mathsf{At}' \subseteq \mathsf{At}} S_{\mathsf{At}'}$.

Canonical construction:

For every $\mathrm{At}' \subseteq \mathsf{At}$, let $S_{\mathrm{At}'}$ be the set of maximally consistent sets $\omega_{\mathrm{At}'}$ of formulae in the sublanguage $L_I^{k,a}(\mathrm{At}')$.

$\langle\{S_{\mathrm{At}'}\}_{\mathsf{At}'\subseteq\mathsf{At}}, \preceq\rangle$ is a complete lattice of disjoint spaces with the order defined by $S_{\mathrm{At}''} \succeq S_{\mathsf{At}'}$ if and only if $\mathsf{At}'' \supseteq \mathsf{At}'$.

$\Omega := \bigcup_{\mathsf{At}'\subseteq\mathsf{At}} S_{\mathsf{At}'}$.

For any $S_{\mathrm{At}''} \succeq S_{\mathrm{At}'}$, surjective projections $r^{\mathrm{At}''}_{\mathrm{At}'} : S_{\mathrm{At}''} \longrightarrow S_{\mathrm{At}'}$ are defined by $r^{\mathrm{At}''}_{\mathrm{At}'}(\omega) := \omega \cap L_I^{k,a}(\mathrm{At}')$.

Theorem For every ω and $i \in I$, the possibility correspondence defined by $\Pi_i(\omega)$

$$:= \left\{ \omega' \in \Omega : \text{ For every formula } \varphi, \begin{array}{l} \text{(i) } k_i\varphi \text{ implies } \varphi \in \omega', \text{ and} \\ \text{(ii) } a_i\varphi \in \omega \text{ iff } \varphi \in \omega' \text{ or } \neg\varphi \in \omega' \end{array} \right\}$$

satisfies Confinement, Generalized Reflexivity, Projections Preserve Ignorance, and Projections Preserve Knowledge. Moreover, for every formula φ, the set of states $[\varphi] := \{\omega \in \Omega : \varphi \in \omega\}$ is a $S_{\mathrm{At}(\varphi)}$-based event, and $[\neg\varphi] = \neg[\varphi]$, $[\varphi \wedge \psi] = [\varphi] \cap [\psi]$, $[k_i\varphi] = K_i[\varphi]$, $[a_i\varphi] = A_i[\varphi]$, and $[u_i\varphi] = U_i[\varphi]$.

Set of events in unawareness structures Σ

Valuation $V : \mathrm{At} \longrightarrow \Sigma$

Set of events in unawareness structures Σ

Valuation $V : \mathrm{At} \longrightarrow \Sigma$

Denote $M = \langle\langle \mathcal{S}, \preceq\rangle, (r_S^{S'})_{S' \preceq S}, (\Pi_i)_{i \in I}, V\rangle$.

Set of events in unawareness structures Σ

Valuation $V : \mathrm{At} \longrightarrow \Sigma$

Denote $M = \langle \langle \mathcal{S}, \preceq \rangle, (r_S^{S'})_{S' \preceq S}, (\Pi_i)_{i \in I}, V \rangle$.

Define inductively the satisfaction relation on the structure of formulae in $L_I^{k,a}(\mathrm{At})$

$$M, \omega \models \top, \text{ for all } \omega \in \Omega,$$

$$M, \omega \models p \text{ if and only if } \omega \in V(p),$$

$$M, \omega \models \varphi \wedge \psi \text{ if and only if } \omega \in [\varphi] \cap [\psi],$$

$$M, \omega \models \neg \varphi \text{ if and only if } \omega \in [\neg \varphi],$$

$$M, \omega \models k_i \varphi \text{ if and only if } \omega \in K_i[\varphi],$$

where $[\varphi] := \{\omega' \in \Omega : M, \omega' \models \varphi\}$, for every formula φ.

In a Kripke structure, a formula is valid if it is true in every state. In an unawareness structure, a formula may not be defined in every state!

In a Kripke structure, a formula is valid if it is true in every state. In an unawareness structure, a formula may not be defined in every state!

A formula φ *is defined in state* ω in M if $\omega \in \bigcap_{p \in \mathsf{At}(\varphi)} (V(p) \cup \neg V(p))$.

A formula φ is *valid* in M if $M, \omega \models \varphi$ for all ω in which φ is defined.

A formula φ is valid if it is valid in all M.

An axiom system is *sound* for a language L with respect to a class of structures if every formula in L that is provable in the axiom system is valid with respect to every structure M.

An axiom system is *complete* for a language L with respect to a class of structures if every formula in L that is valid in every structure M is provable in the axiom system.

An axiom system is *sound* for a language L with respect to a class of structures if every formula in L that is provable in the axiom system is valid with respect to every structure M.

An axiom system is *complete* for a language L with respect to a class of structures if every formula in L that is valid in every structure M is provable in the axiom system.

Theorem For the language $L_I^{k,a}(\mathrm{At})$, the axiom system $S5_I^{k,a}$ is a sound and complete axiomatization with respect to unawareness structures.

An Example: Speculative Trade

How to model the following example?

Two agents: An owner *o* of a firm and a potential buyer *b*.

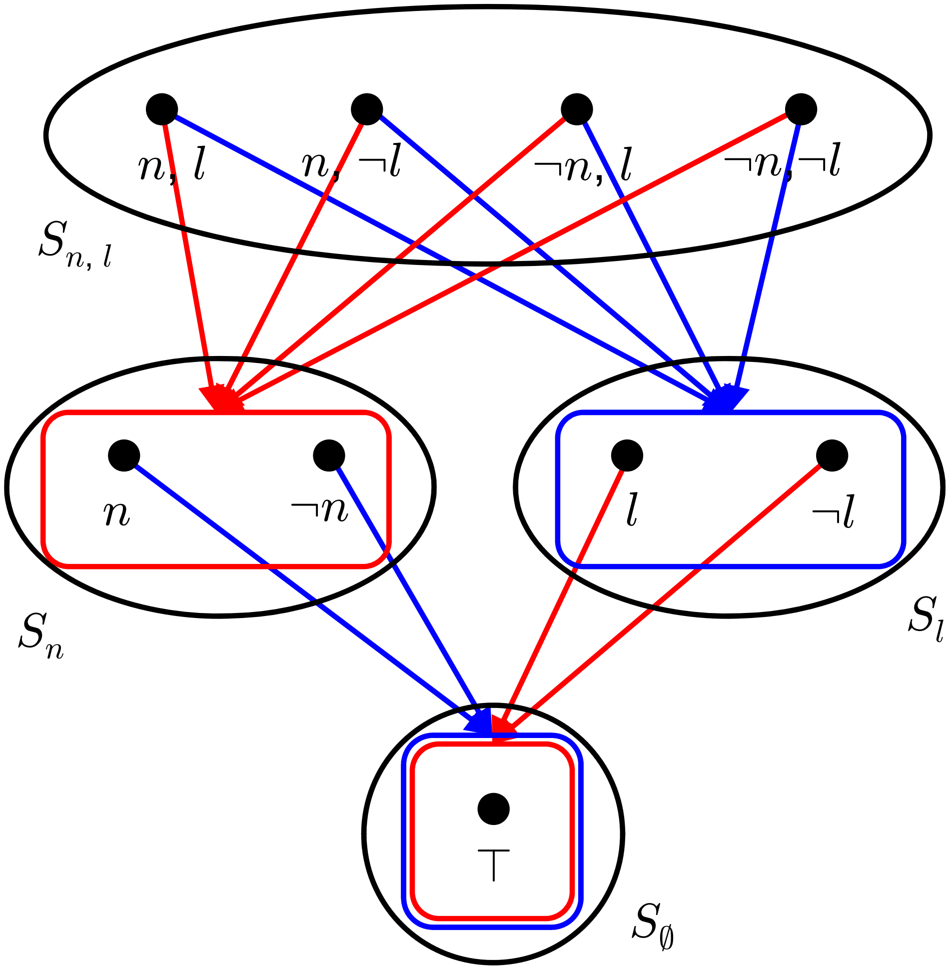
Status quo value of the firm is $100.

The owner is aware of a potential **lawsuit** l that may cost the firm $20. The buyer is unaware of it. The owner knows that the buyer is unaware of it.

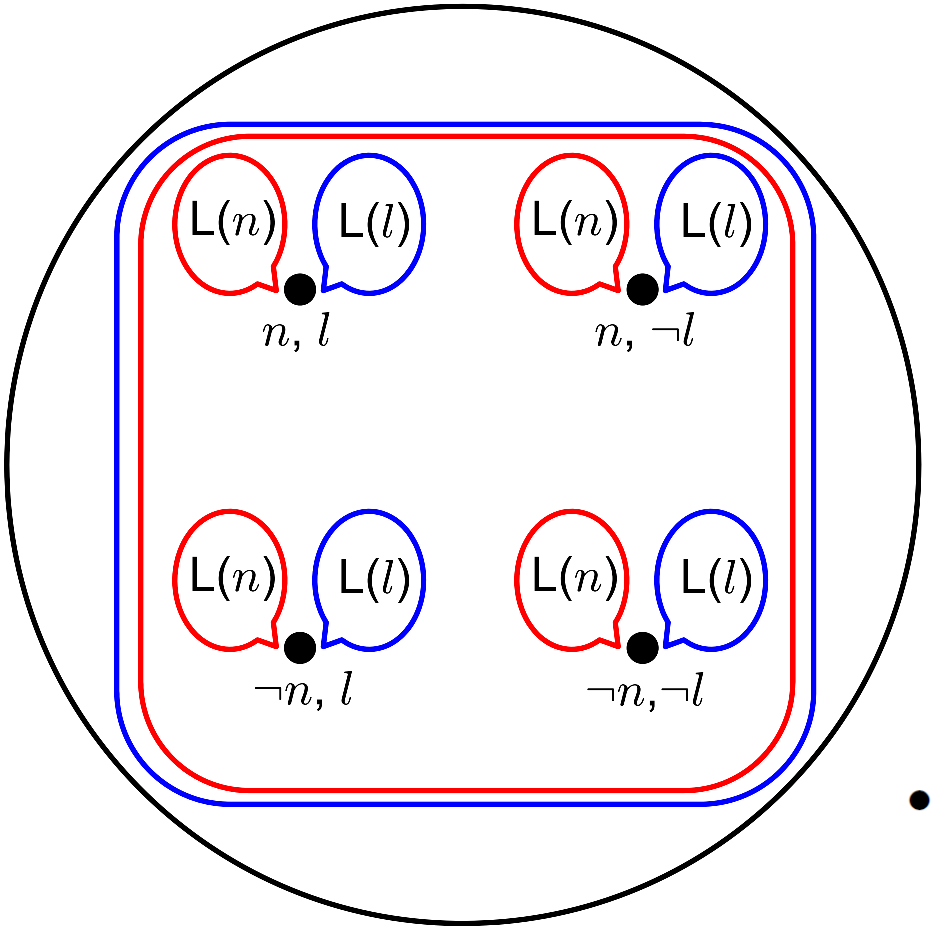
The buyer is aware of a potential **innovation** n that may enhance the value of the firm by $20. The owner is unaware of it. The buyer knows that the owner is unaware of it.

Suppose the buyer is offering to buy the firm for $100. Is the owner going to sell to her?

An Example: Speculative Trade



The First Approach: Awareness Structures
(Fagin and Halpern, Artificial Intelligence 1988)



Language $\mathsf{L}_n^{K,L,A}(\mathsf{At})$ defined by

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \psi \mid K_i\varphi \mid L_i\varphi \mid A_i\varphi$$

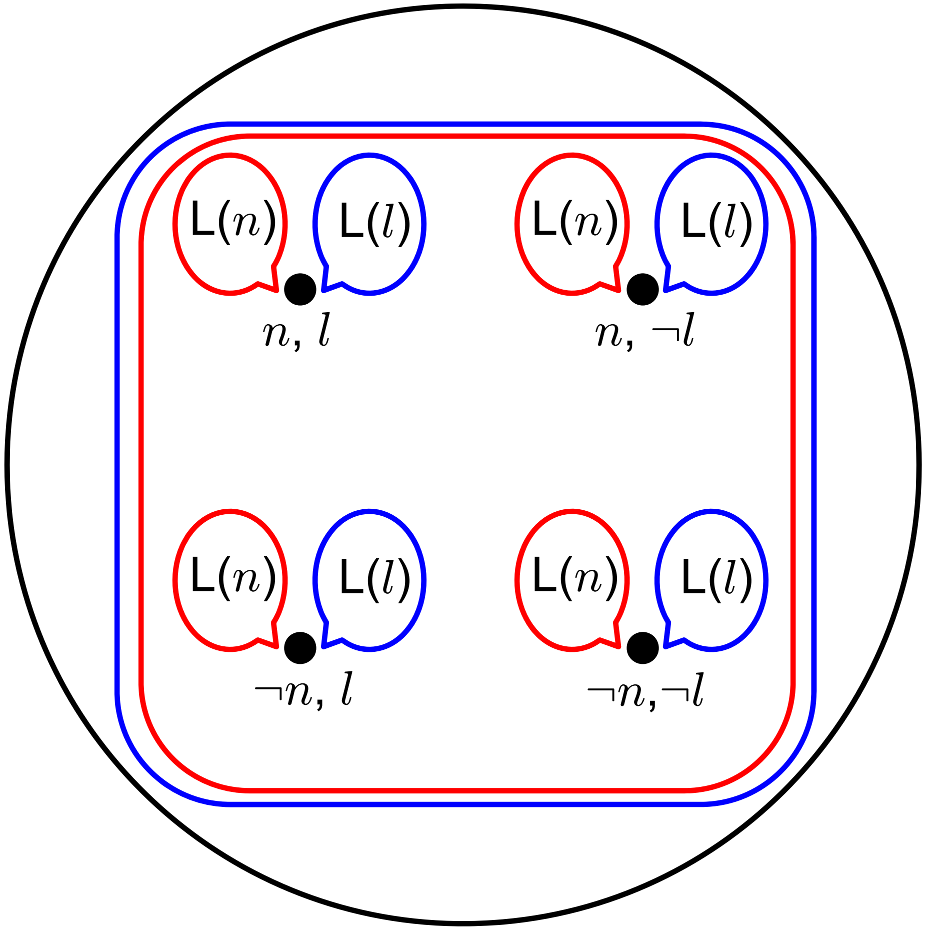
An awareness structure consists of

- a set of states S,
- accessibility relations $R_i \subseteq S \times S$,
- awareness correspondences
  $\mathcal{A}_i : S \longrightarrow 2^{\mathsf{L}_n^{K,L,A}(\mathsf{At})}$
- a valuation $V : S \times \mathsf{At} \longrightarrow \{true, false\}$

$$M, s \models L_i\varphi \text{ if and only if } M, t \models \varphi \text{ for all } t \in S \text{ such that } (s, t) \in R_i$$

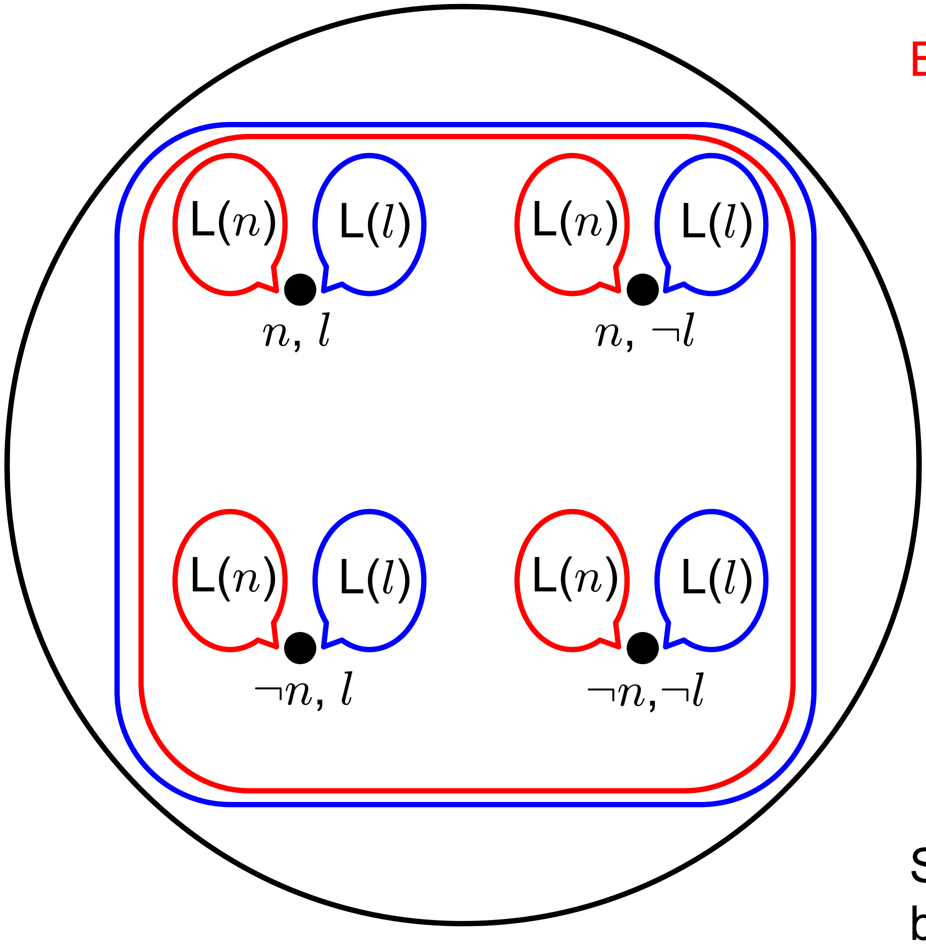
$$M, s \models A_i\varphi \text{ if and only if } \varphi \in \mathcal{A}_i(s)$$

$$M, s \models K_i\varphi \text{ if and only if } M, s \models A_i\varphi \text{ and } M, s \models L_i\varphi.$$



Fagin and Halpern (1988), Halpern (2001), Halpern and Rego (2008) provide sound and complete axiomatizations of awareness structures under various assumptions on accessibility relations and awareness correspondences. (details)

The example models our story.



Buyer: "I (implicitly) know that the owner is aware of the lawsuit but unfortunately I am unaware of the lawsuit."

It models reasoning of an outside observer about the knowledge and awareness of agents but not the agents' subjective reasoning that economists, decision theorists, and game theorists are interested in.

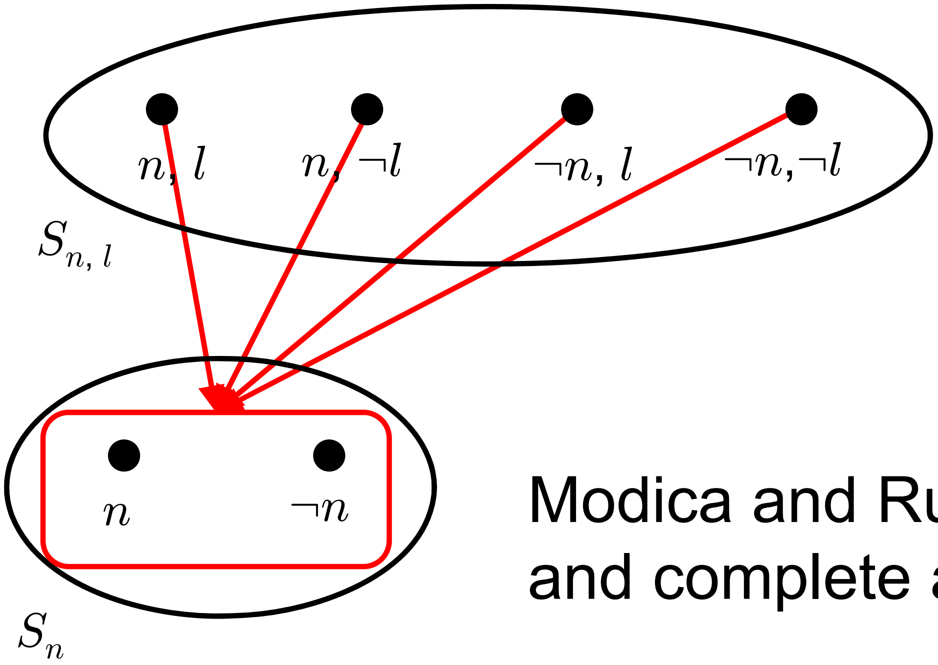
Semantics is not syntax-free. What would be the analogue to a Kripke frame?

Awareness structures are very flexible. For instance,

$$\varphi \wedge \psi \in \mathcal{A}_i(s) \text{ but } \psi \wedge \varphi \notin \mathcal{A}_i(s)$$

Should be very useful to model various forms of framing.

The Second Approach: Generalized Standard Models
(Modica and Rustichini, GEB 1999)



Single-agent structure

Modica-Rustichini definition

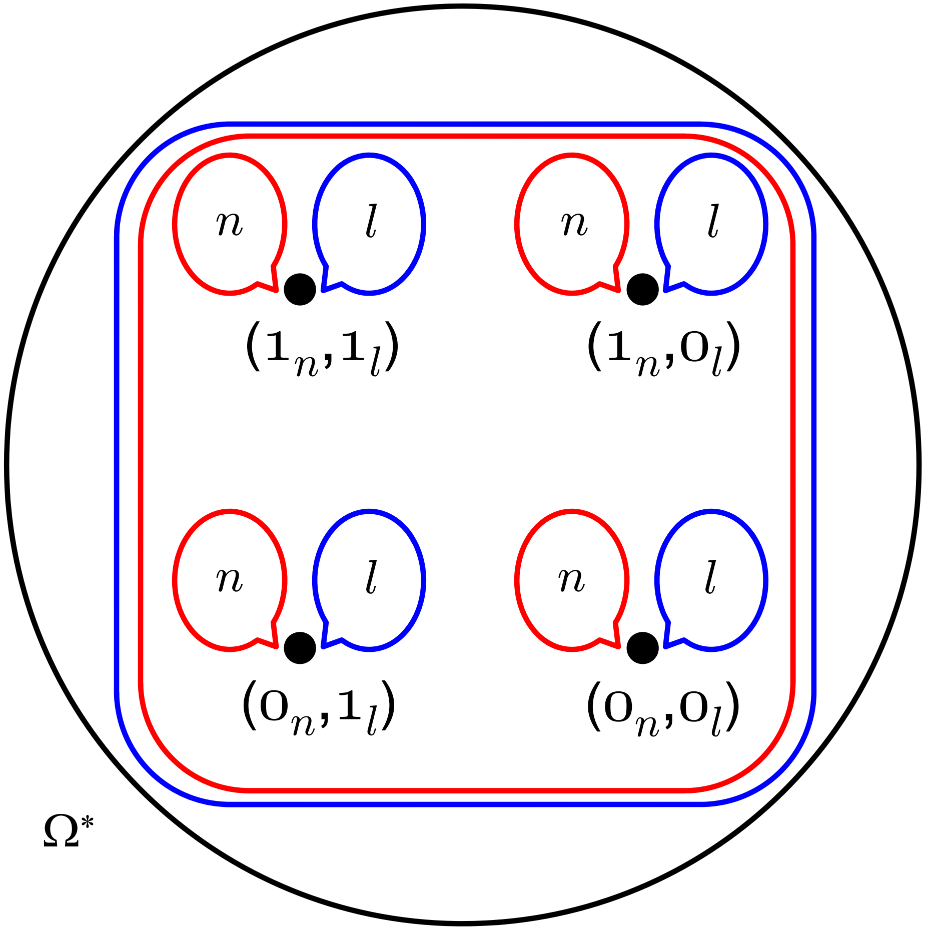
$$A_i\varphi := K_i\varphi \vee K_i\neg K_i\varphi$$

Modica and Rustichini (1999) prove a sound and complete axiomatization.

Use “objective” validity.

Models reasoning of an outside observer about knowledge and awareness of a single agent.

A Fourth Approach: Product Models
(Li, 2008, JET 2009)



A product model consists of

- a set of questions Q^*
- a set of objective states

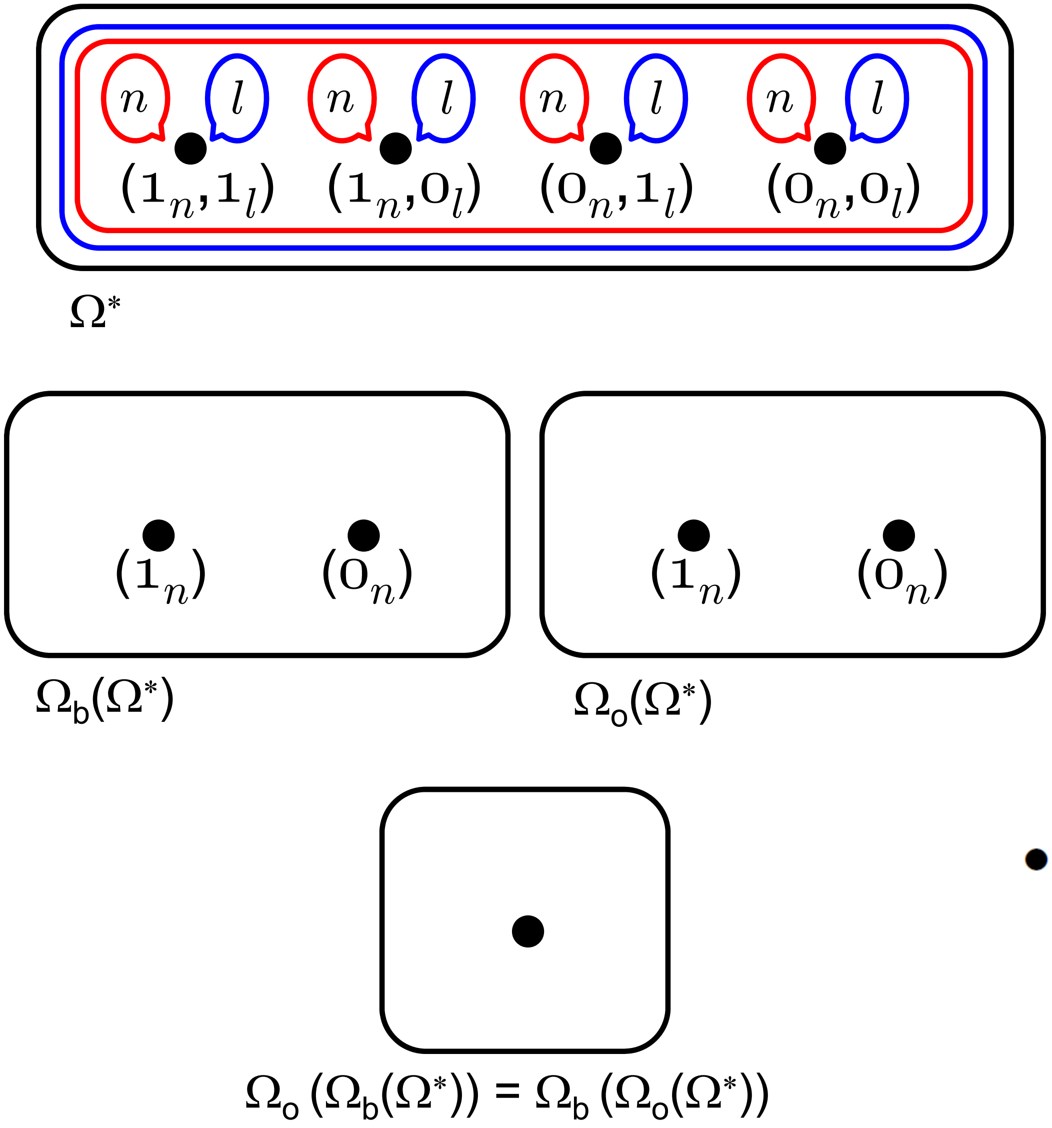
  $\Omega^* := \times_{q \in Q^*} \{1_q, 0_q\}$
- an awareness correspondence

  $\mathcal{A}_i : \Omega^* \to 2^{Q^*}$
- a possibility correspondence

  $\Pi_i : \Omega^* \to 2^{\Omega^*}$

A Fourth Approach: Product Models

(Li, 2008, JET 2009)



A product model consists of

- a set of questions Q^*
- a set of objective states

  $$\Omega^* := \times_{q \in Q^*} \{1_q, 0_q\}$$

- an awareness correspondence

  $$\mathcal{A}_i : \Omega^* \to 2^{Q^*}$$

- a possibility correspondence

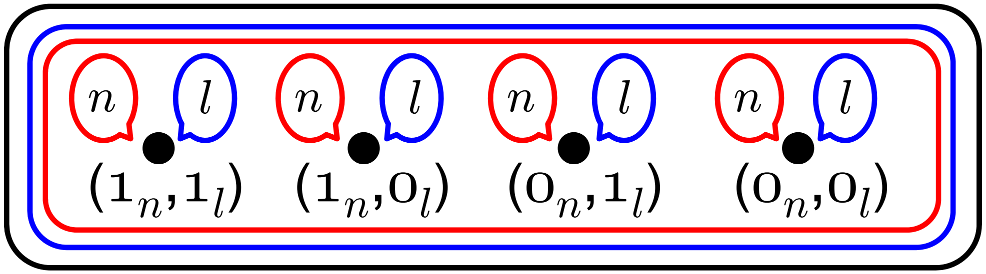
  $$\Pi_i : \Omega^* \to 2^{\Omega^*}$$

- subjective state-spaces

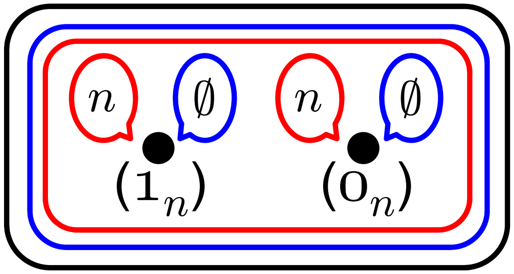
  $$\Omega_i(\omega) := \times_{q \in \mathcal{A}_i(\omega)} \{1_q, 0_q\}$$

A Fourth Approach: Product Models

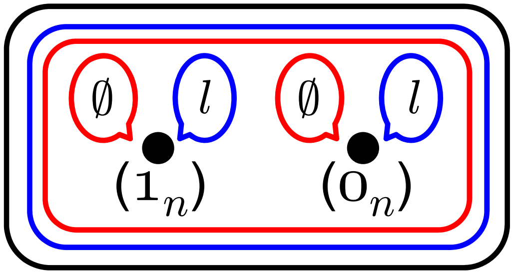
(Li, 2008, JET 2009)



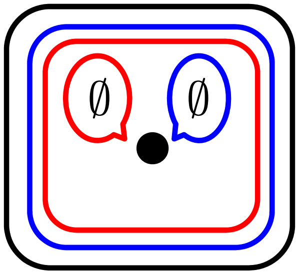
Ω^*



$\Omega_b(\Omega^*)$



$\Omega_o(\Omega^*)$



$\Omega_o\,(\Omega_b(\Omega^*)) = \Omega_b\,(\Omega_o(\Omega^*))$

A product model consists of

- a set of questions Q^*
- a set of objective states

  $$\Omega^* := \times_{q \in Q^*} \{1_q, 0_q\}$$

- an awareness correspondence

  $$\mathcal{A}_i : \Omega^* \to 2^{Q^*}$$

- a possibility correspondence

  $$\Pi_i : \Omega^* \to 2^{\Omega^*}$$

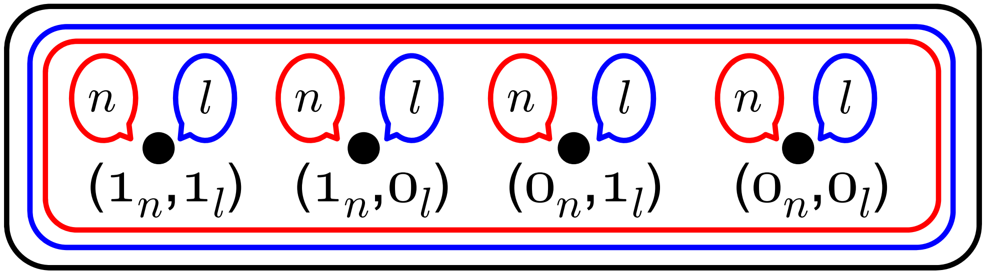
- subjective state-spaces

  $$\Omega_i(\omega) := \times_{q \in \mathcal{A}_i(\omega)} \{1_q, 0_q\}$$

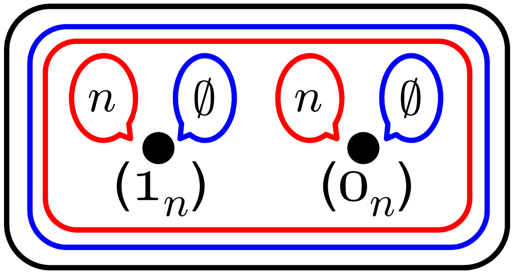
- “subjective” versions of $\mathcal{A}_i$ and Π_i

A Fourth Approach: Product Models

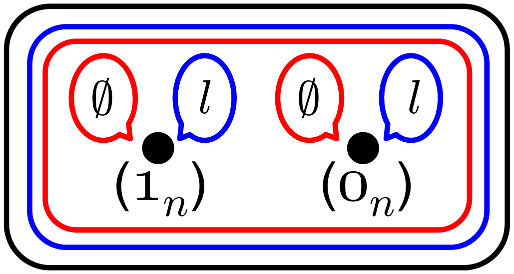
(Li, 2008, JET 2009)



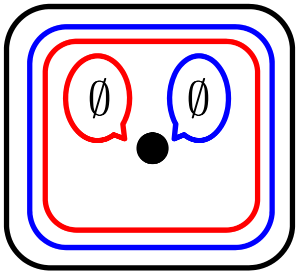
Ω^*



$\Omega_b(\Omega^*)$



$\Omega_o(\Omega^*)$



$\Omega_o (\Omega_b(\Omega^*)) = \Omega_b (\Omega_o(\Omega^*))$

Example models our story

Possibility correspondence models implicit knowledge.

Sublattices represent subjective perceptions of the situation, while the objective state space represents the outside modeler's view.

Li (2009) proves properties of unawareness and knowledge

Further Epistemic Approaches

Awareness of Unawareness:

- Awareness structures with propositional quantifiers: Halpern and Rego (MSS 2013), Halpern and Rego (GEB 2009)
- First-order modal logic with unawareness of objects: Board and Chung (2011)
- First-order modal logic with unawareness of objects and properties: Quantified awareness neighborhood structures: Sillari (RSL 2008)
- Awareness of unawareness without quantifiers: Agotnes and Alechina (2007), Walker (MSS 2014)

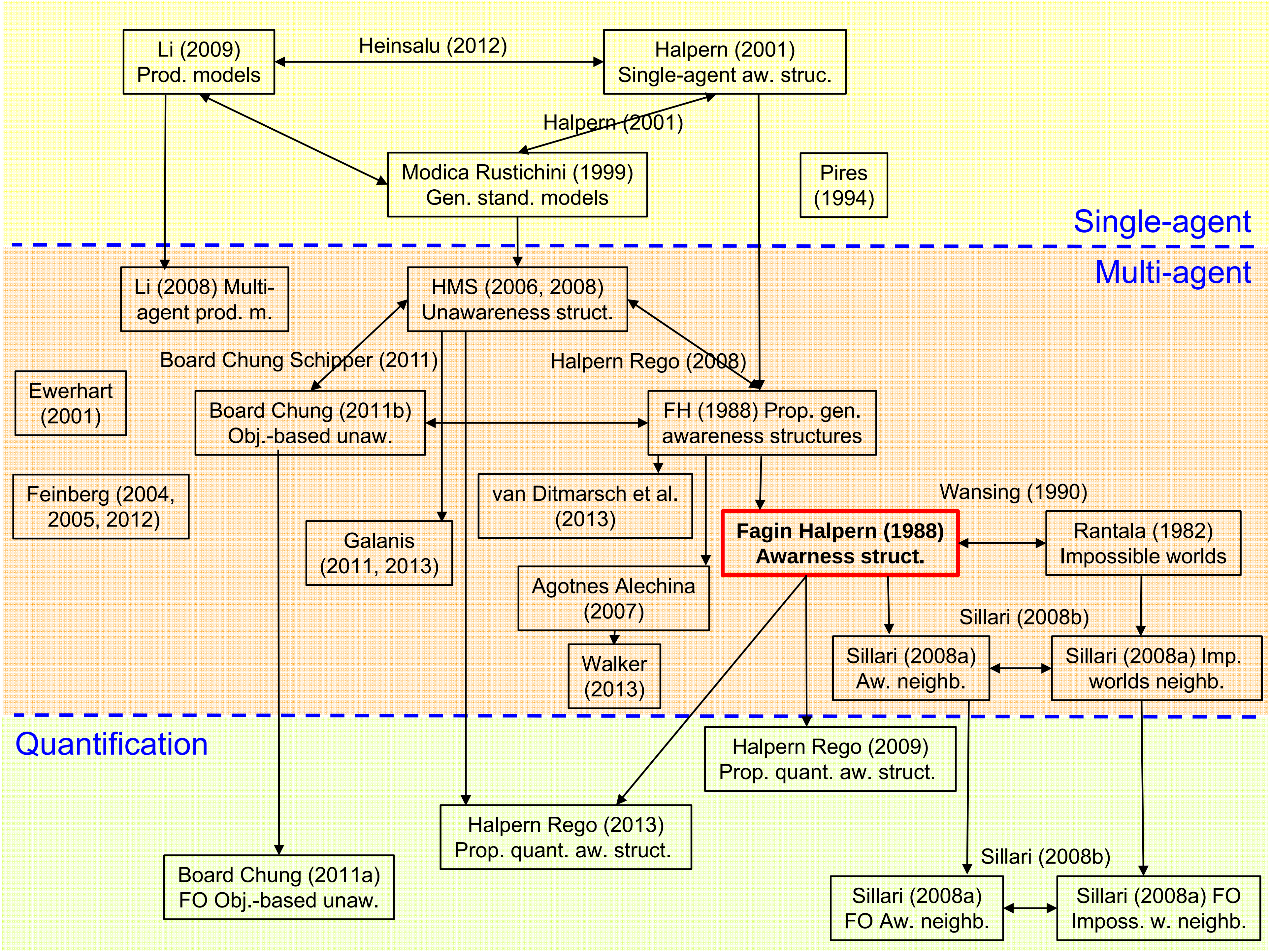
Dynamic awareness:

…

"Evolution" of the literature on epistemic models of unawareness

Survey:

Schipper, B.C. *"Awareness",* in: Handbook of Epistemic Logic, Chapter 3, H. van Ditmarsch, J.Y. Halpern, W. van der Hoek and B. Kooi (Eds.), College Publications, London, 2015, 77–146.



Single-agent

Li (2009)
Prod. models

Heinsalu (2012)

Halpern (2001)
Single-agent aw. struc.

Halpern (2001)

Modica Rustichini (1999)
Gen. stand. models

Pires
(1994)

Multi-agent

Li (2008) Multi-agent prod. m.

HMS (2006, 2008)
Unawareness struct.

Board Chung Schipper (2011)

Halpern Rego (2008)

Ewerhart
(2001)

Board Chung (2011b)
Obj.-based unaw.

FH (1988) Prop. gen.
awareness structures

Feinberg (2004,
2005, 2012)

Galanis
(2011, 2013)

van Ditmarsch et al.
(2013)

Wansing (1990)

Fagin Halpern (1988)
Awarness struct.

Rantala (1982)
Impossible worlds

Agotnes Alechina
(2007)

Walker
(2013)

Sillari (2008b)

Sillari (2008a)
Aw. neighb.

Sillari (2008a) Imp.
worlds neighb.

Quantification

Halpern Rego (2009)
Prop. quant. aw. struct.

Halpern Rego (2013)
Prop. quant. aw. struct.

Sillari (2008b)

Board Chung (2011a)
FO Obj.-based unaw.

Sillari (2008a)
FO Aw. neighb.

Sillari (2008a) FO
Imposs. w. neighb.

The Unawareness Bibliography

Burkhard C. Schipper

If you want to have your work included or something needs updating, please email me.

Conferences

International Workshop on 'Unawareness', January 29-30, 2014, University of Queensland, Organizer: Simon Grant, Jeff Kline, and John Quiggin

Unawareness: Conceptualization and Modeling, October 29, 2011, Johns Hopkins University, Organizer: Edi Karni

Papers

Agotnes, T. and N. Alechina (2007). Full and relative awareness: A decidable logic for reasoning about knowledge of unawareness, in: D. Samet (Ed.), Proceedings of the 11th Conference on Theoretical Aspects of Rationality and Knowledge (TARK 2007), Presses Universitaires De Louvain, 6-14.

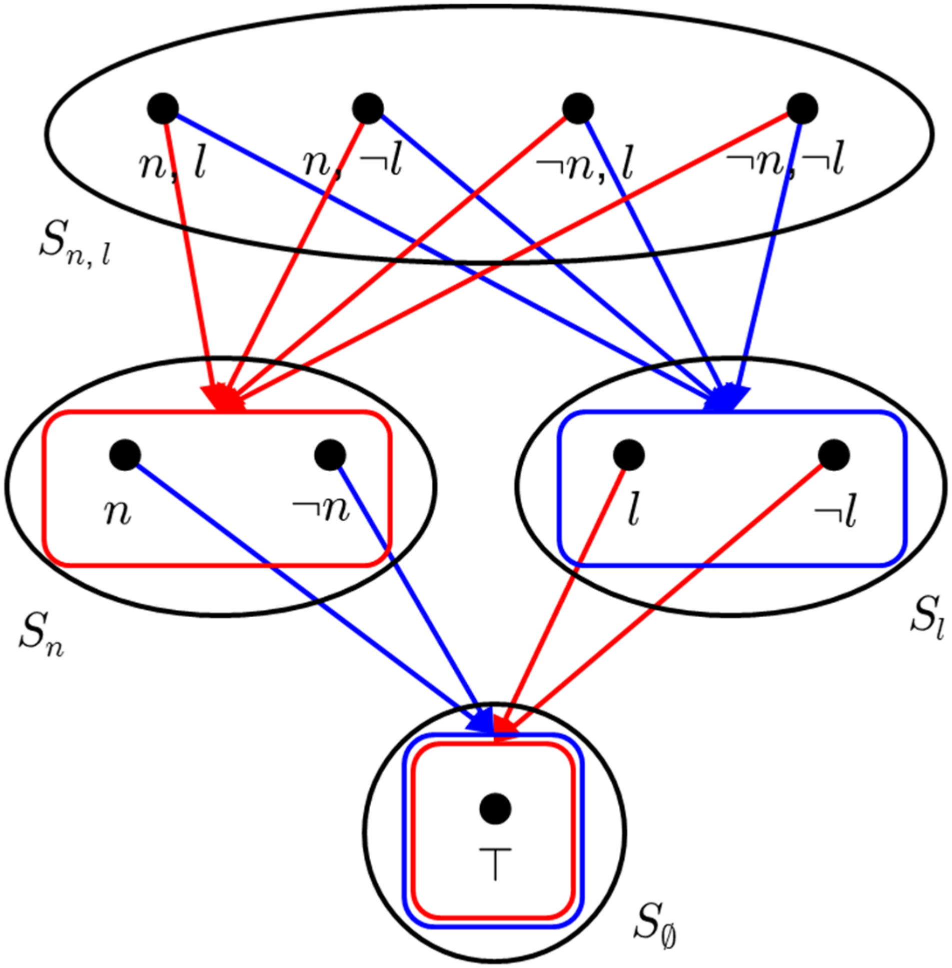
Auster, S. (2013). Asymmetric awareness and moral hazard, *Games and Economic Behavior*, 82, 503-521.

van Benthem, J. and Velazquez-Quesada, F.R. (2010). The dynamic of awareness, *Synthese*, 177, 5-27.

Board, O. and K.S. Chung (2011). Object-based unawareness: Axioms, an earlier version published in: G. Bonanno, W. van der Hoek, and M. Wooldridge (Eds.), Proceedings of

Returning to the speculative trade example:
Say that at a state s an agent is willing to trade at the price $\$x$ if either she strictly prefers to trade at $\$x$ or she is indifferent between trading or not at $\$x$.

- At \$100, both agents are willing to trade (in every state).
- This is common knowledge among the agents at all states.
- Yet, each agent has a strict preference to trade in all in all states except the lowest space. Both are indifferent between trading and not trading in the lowest space. The states in the lowest space are the “states of mind” of an agent as viewed by the other agent.



Returning to the speculative trade example:

Say that at a state s an agent is willing to trade at the price $\$x$ if either she strictly prefers to trade at $\$x$ or she is indifferent between trading or not at $\$x$.

- At $100, both agents are willing to trade (in every state).
- This is common knowledge among the agents at all states.
- Yet, each agent has a strict preference to trade in all in all states except the lowest space. Both are indifferent between trading and not trading in the lowest space. The states in the lowest space are the "states of mind" of an agent as viewed by the other agent.

This is in contrast to the "No-speculative-trade-theorems" of structures without unawareness (e.g., Milgrom and Stokey, 1982):

If there is a common prior and common knowledge of willingness to trade, then both must be indifferent to trade.

Need to talk about probabilities.

Outline

1. Informal introduction
2. Epistemic models of unawareness
3. Type spaces with unawareness
4. Speculation
5. Bayesian games with unawareness
6. Revealed unawareness
7. Dynamic games with unawareness
8. ...(???)

3. Type spaces with unawareness

Type Spaces with Unawareness

- Economists frequently make use of probabilistic notions of belief
- Quantified notions of belief are often useful for applications (optimization under uncertainty)
- How to model unawareness and probabilistic beliefs?

Each $S \in \mathcal{S}$ is now a measurable space with sigma-field $\mathcal{F}_S$.

If $S' \succeq S$, we require a measurable surjective projection $r_S^{S'} : S' \to S$.

Σ denotes now the set of *measurable* events in the unawareness structure.

Denote by $\triangle(S)$ is a set of probability measures on $(S, \mathcal{F}_S)$ endowed with the sigma-field $\mathcal{F}_{\triangle(S)}$ generated by $\{\mu \in \triangle(S) : \mu(D) \geq p\}$, $D \in \mathcal{F}_S$, $p \in [0,1]$.

For $\mu \in \triangle(S')$, the marginal $\mu_{|S}$ of μ on $S \preceq S'$ is defined by

$$\mu_{|S}\left(D\right) := \mu\left(\left(r_S^{S'}\right)^{-1}\left(D\right)\right), \quad D \in \mathcal{F}_S.$$

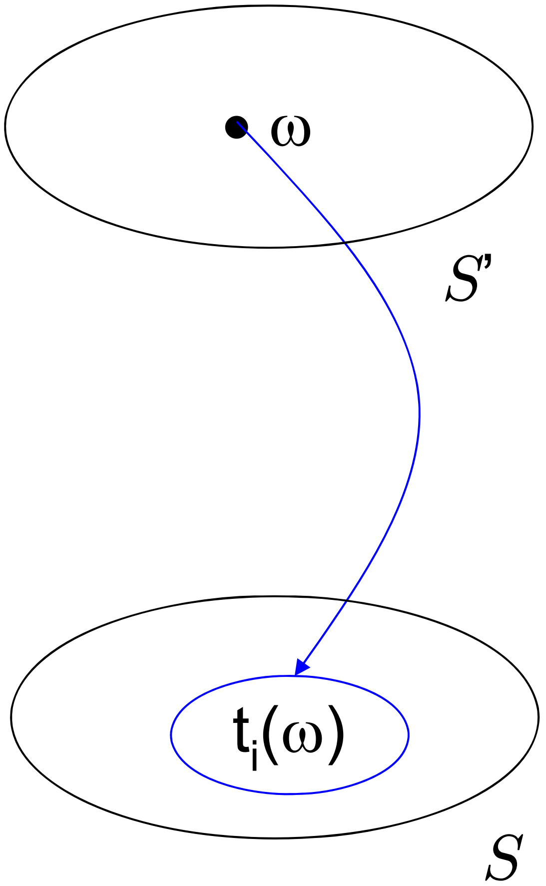
Denote by S_μ the space on which μ is a probability measure.

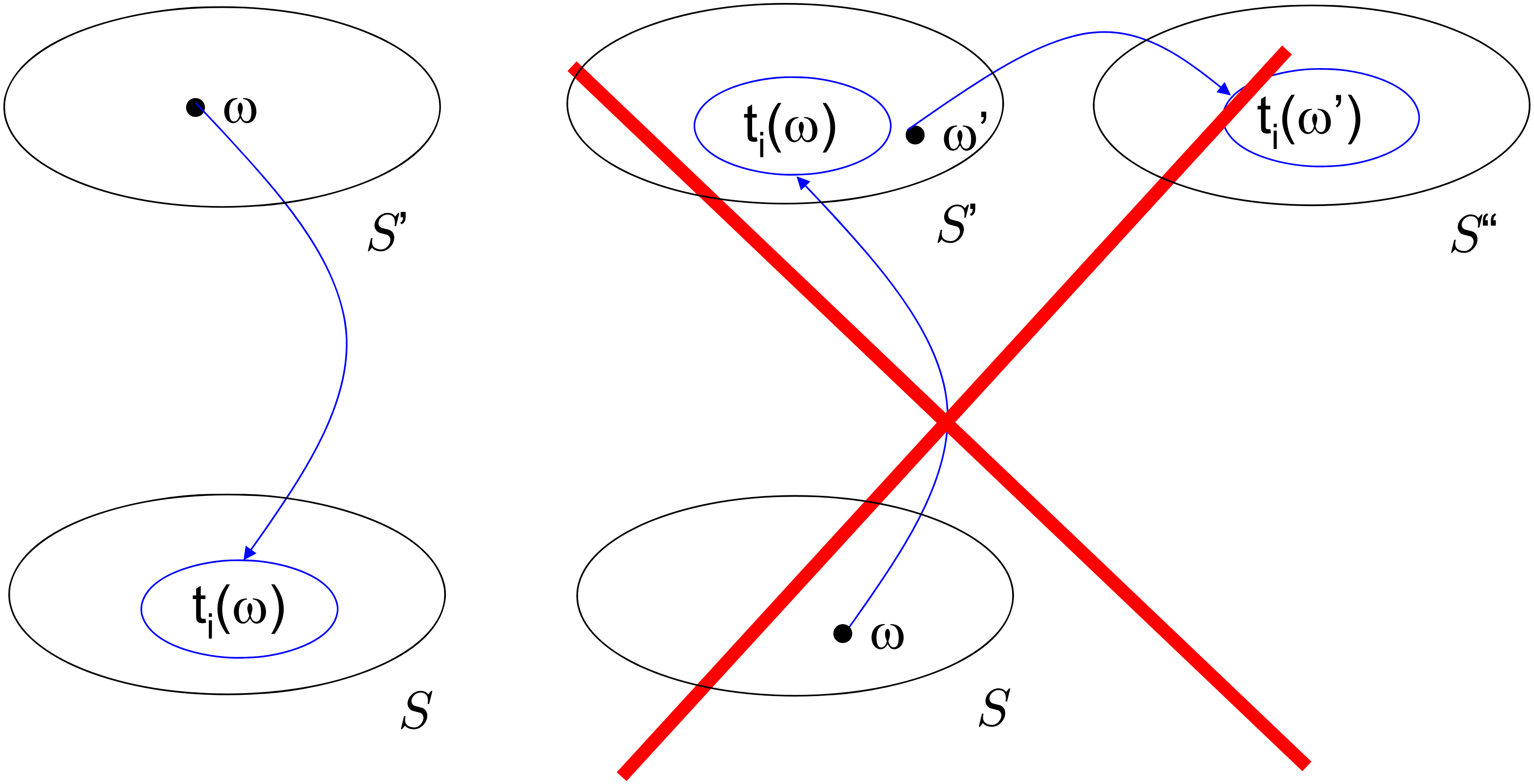
Whenever $S_\mu \succeq S(E)$, we abuse notation $\mu(E) = \mu(E \cap S_\mu)$. If $S_\mu \not\succeq S(E)$, then $\mu(E)$ is undefined.

Definition For each individual $i \in I$ there is a *type mapping* $t_i : \Omega \to \bigcup_{\alpha \in \mathcal{A}} \Delta(S_\alpha)$, which is measurable in the sense that for every $S \in \mathcal{S}$ and $Q \in \mathcal{F}_{\Delta(S)}$ we have $t_i^{-1}(Q) \cap S \in \mathcal{F}_S$, for all $S \in \mathcal{S}$.

We require the type mapping t_i to satisfy the following properties:

(0) *Confinement:* If $\omega \in S'$ then $t_i(\omega) \in \Delta(S)$ for some $S \preceq S'$.


ω
S'
$t_i(\omega)$
S

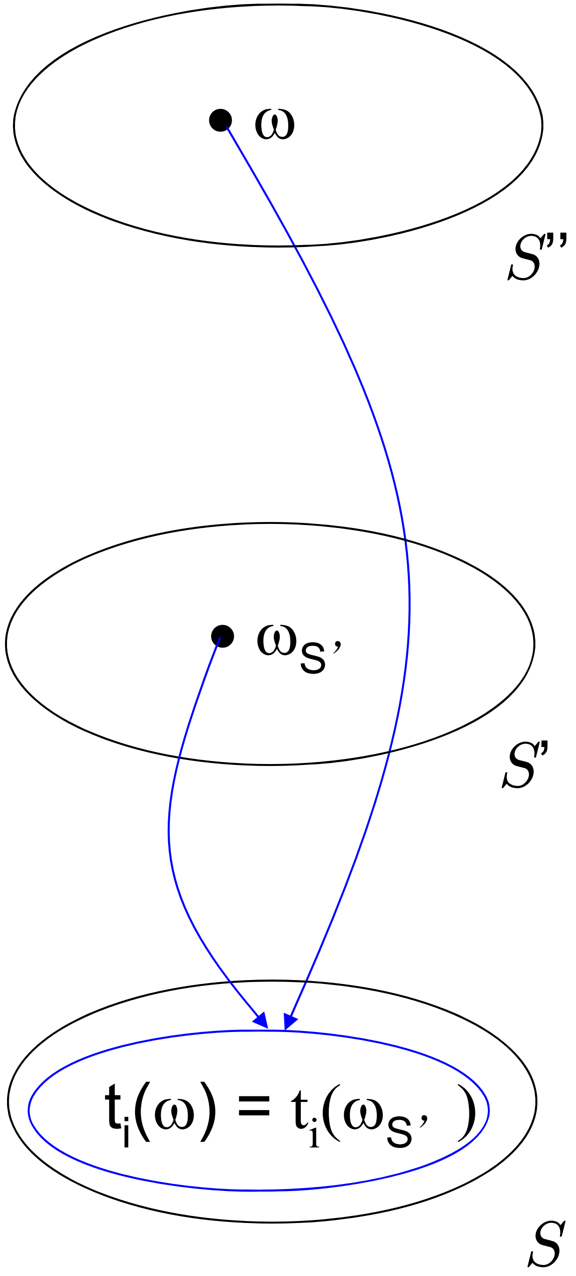

ω
S'
$t_i(\omega)$
S
$t_i(\omega)$
ω'
$t_i(\omega')$
S'
S''
ω
S

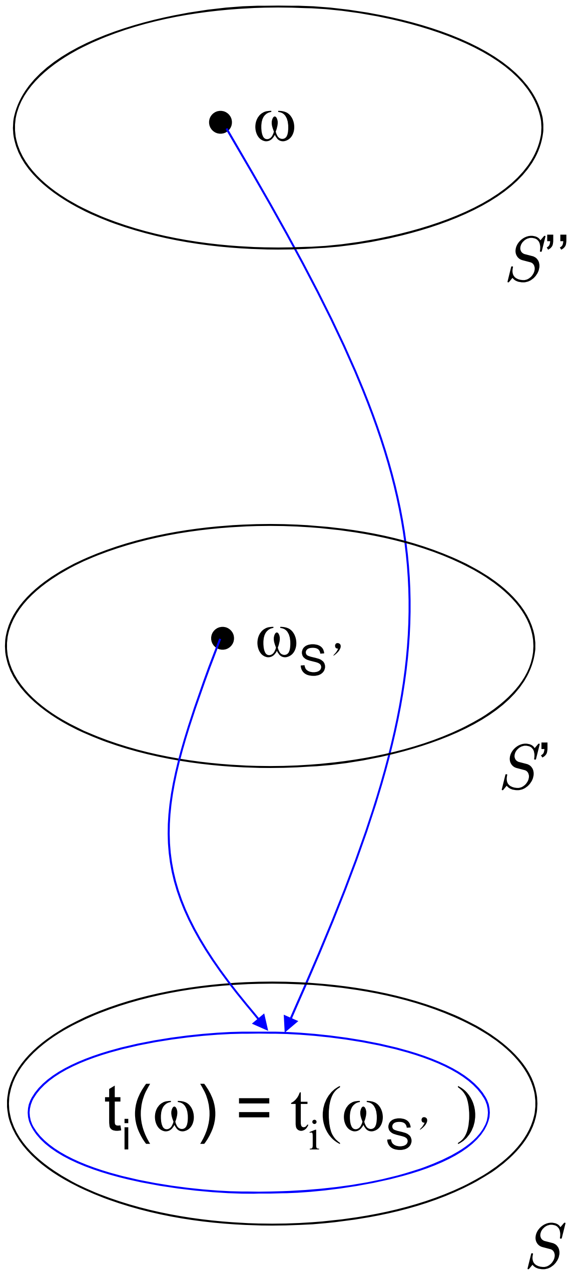
Definition For each individual $i \in I$ there is a *type mapping* $t_i : \Omega \to \bigcup_{\alpha \in \mathcal{A}} \Delta\left(S_\alpha\right)$, which is measurable in the sense that for every $S \in \mathcal{S}$ and $Q \in \mathcal{F}_{\Delta(S)}$ we have $t_i^{-1}(Q) \cap S \in \mathcal{F}_S$, for all $S \in \mathcal{S}$.

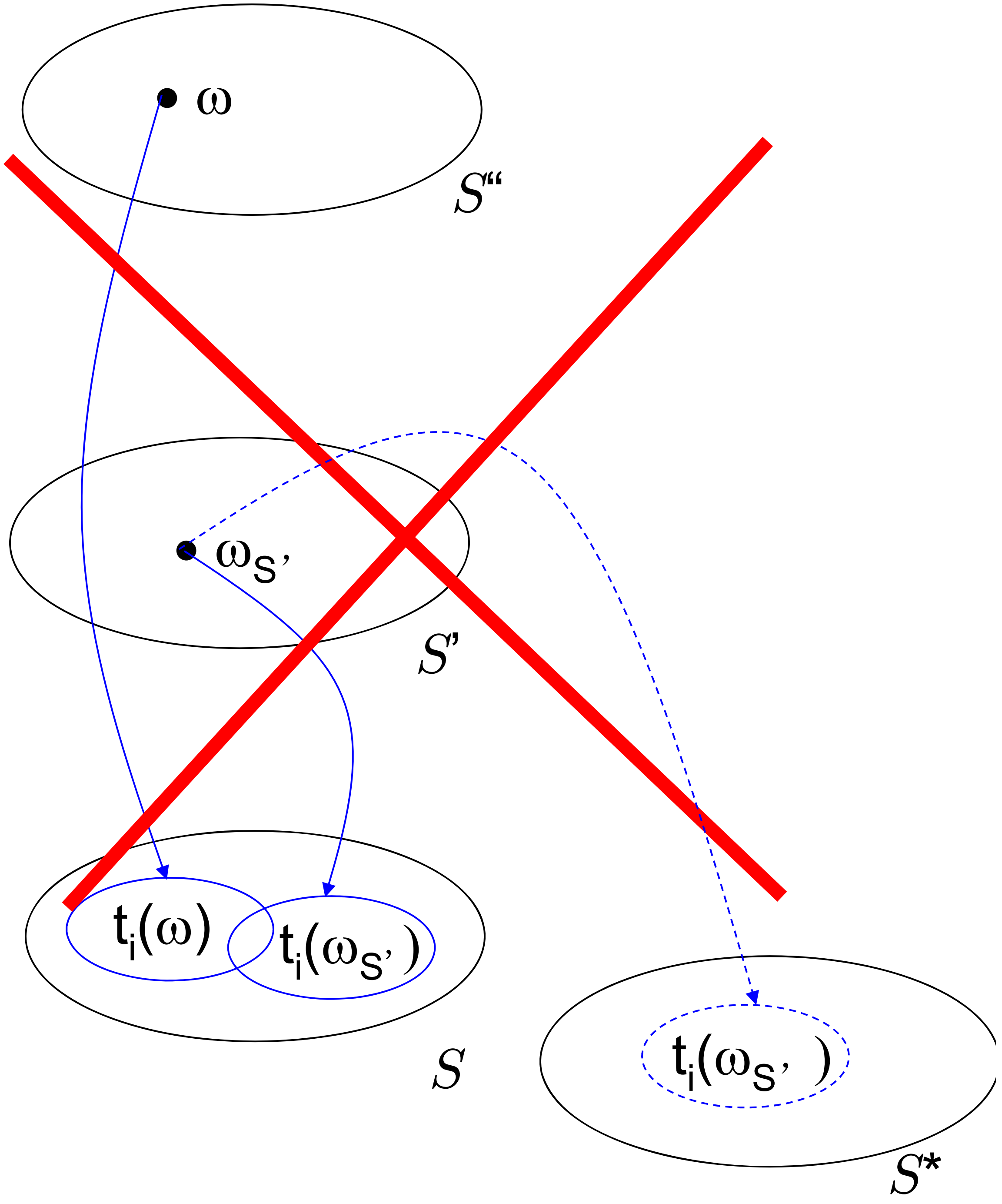
We require the type mapping t_i to satisfy the following properties:

(0) *Confinement:* If $\omega \in S'$ then $t_i(\omega) \in \Delta\left(S\right)$ for some $S \preceq S'$.

(1) If $S'' \succeq S' \succeq S$, $\omega \in S''$, and $t_i(\omega) \in \triangle(S)$ then $t_i(\omega_{S'}) = t_i(\omega)$.


ω
S''
ω_{S'}
S'
t_i(ω) = t_i(ω_{S'})
S


ω
S''
$\omega_{S'}$
S'
$t_i(\omega) = t_i(\omega_{S'})$
S


ω
S''
$\omega_{S'}$
S'
$t_i(\omega)$
$t_i(\omega_{S'})$
S
$t_i(\omega_{S'})$
S*

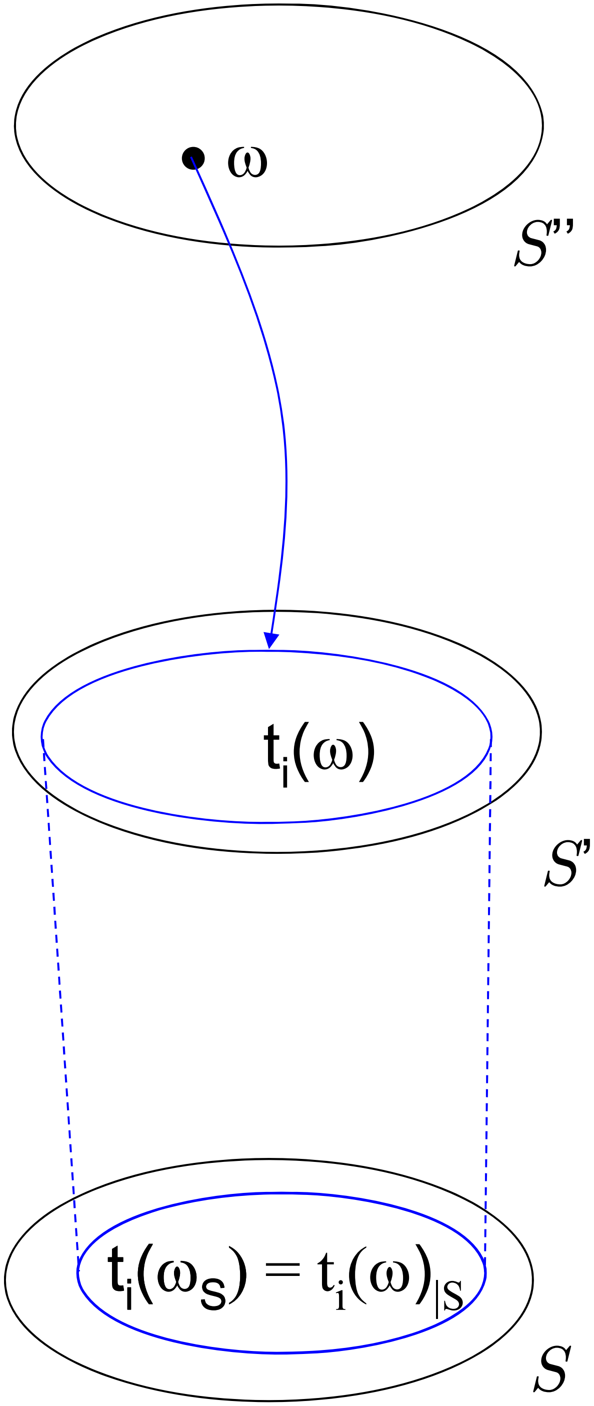
Definition For each individual $i \in I$ there is a *type mapping* $t_i : \Omega \to \bigcup_{\alpha \in \mathcal{A}} \Delta(S_\alpha)$, which is measurable in the sense that for every $S \in \mathcal{S}$ and $Q \in \mathcal{F}_{\Delta(S)}$ we have $t_i^{-1}(Q) \cap S \in \mathcal{F}_S$, for all $S \in \mathcal{S}$.

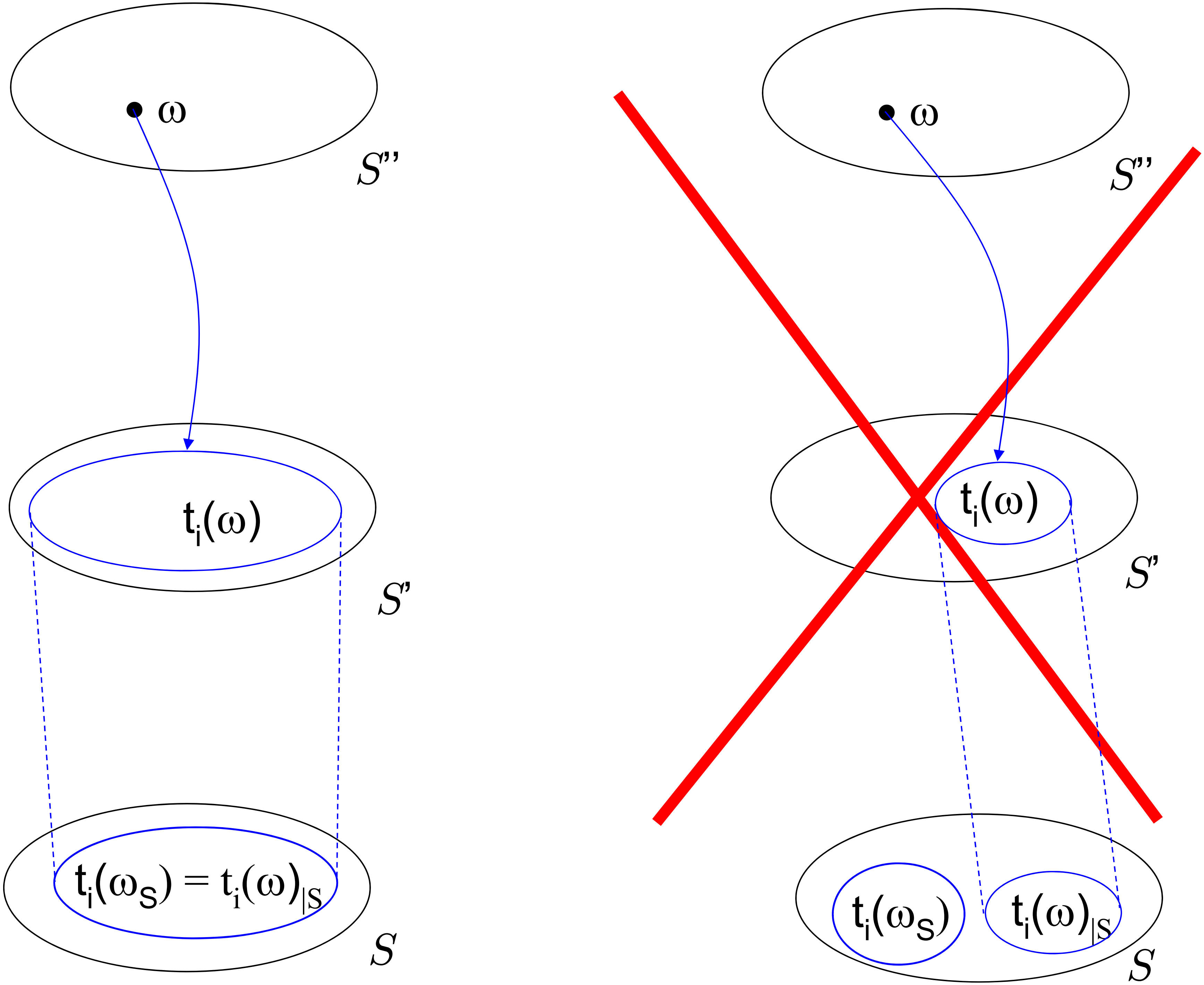
We require the type mapping t_i to satisfy the following properties:

(0) *Confinement:* If $\omega \in S'$ then $t_i(\omega) \in \Delta(S)$ for some $S \preceq S'$.

(1) If $S'' \succeq S' \succeq S$, $\omega \in S''$, and $t_i(\omega) \in \triangle(S)$ then $t_i(\omega_{S'}) = t_i(\omega)$.

(2) If $S'' \succeq S' \succeq S$, $\omega \in S''$, and $t_i(\omega) \in \triangle(S')$ then $t_i(\omega_S) = t_i(\omega)_{|S}$.


ω
S''
$t_i(\omega)$
S'
$t_i(\omega_S) = t_i(\omega)_{|S}$
S


ω
S''
$t_i(\omega)$
S'
$t_i(\omega_S) = t_i(\omega)_{|S}$
S
ω
S''
$t_i(\omega)$
S'
$t_i(\omega_S)$
$t_i(\omega)_{|S}$
S

Definition For each individual $i \in I$ there is a *type mapping* $t_i : \Omega \to \bigcup_{\alpha \in \mathcal{A}} \Delta(S_\alpha)$, which is measurable in the sense that for every $S \in \mathcal{S}$ and $Q \in \mathcal{F}_{\Delta(S)}$ we have $t_i^{-1}(Q) \cap S \in \mathcal{F}_S$, for all $S \in \mathcal{S}$.

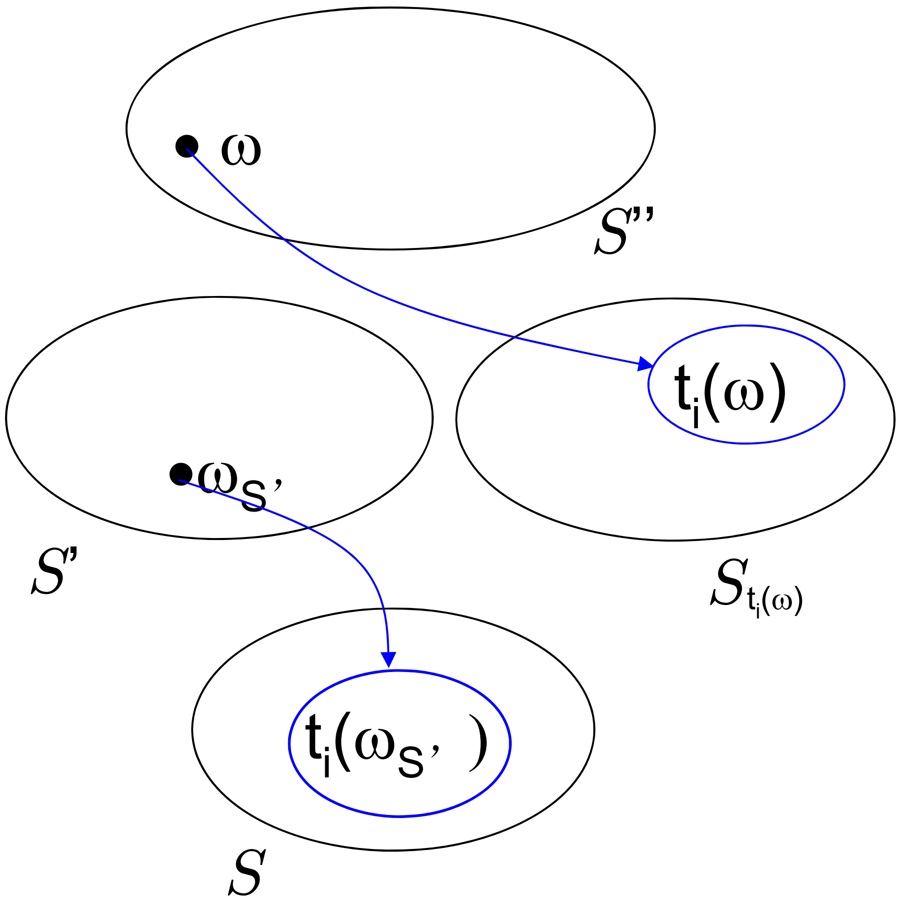
We require the type mapping t_i to satisfy the following properties:

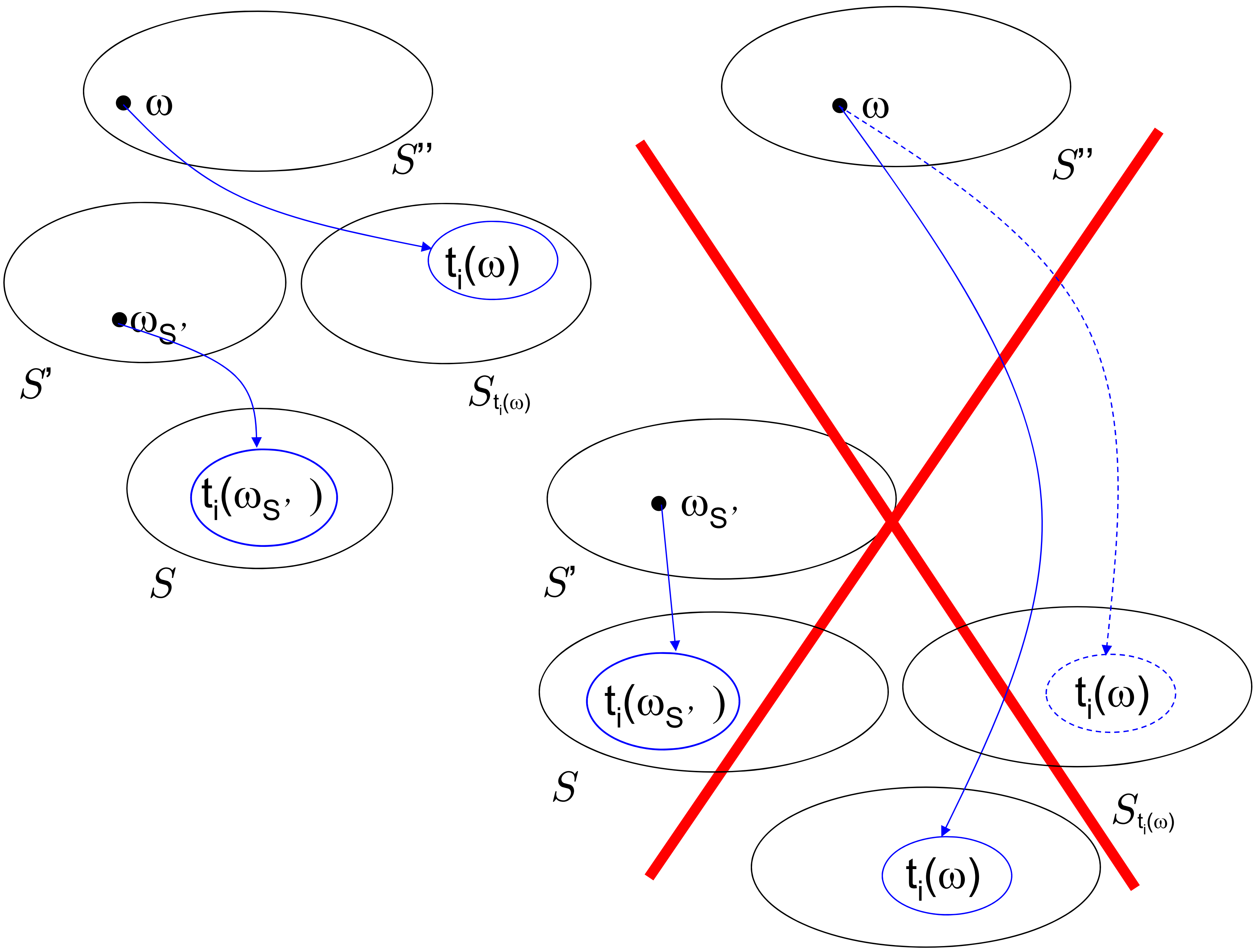
(0) *Confinement:* If $\omega \in S'$ then $t_i(\omega) \in \Delta(S)$ for some $S \preceq S'$.

(1) If $S'' \succeq S' \succeq S$, $\omega \in S''$, and $t_i(\omega) \in \triangle(S)$ then $t_i(\omega_{S'}) = t_i(\omega)$.

(2) If $S'' \succeq S' \succeq S$, $\omega \in S''$, and $t_i(\omega) \in \triangle(S')$ then $t_i(\omega_S) = t_i(\omega)_{|S}$.

(3) If $S'' \succeq S' \succeq S$, $\omega \in S''$, and $t_i(\omega_{S'}) \in \triangle(S)$ then $S_{t_i(\omega)} \succeq S$.


ω
S''
$t_i(\omega)$
$\omega_{S'}$
S'
$S_{t_i(\omega)}$
$t_i(\omega_{S'})$
S


ω
S''
$t_i(\omega)$
$\omega_{S'}$
S'
$S_{t_i(\omega)}$
$t_i(\omega_{S'})$
S
ω
S''
$\omega_{S'}$
S'
$t_i(\omega_{S'})$
S
$t_i(\omega)$
$S_{t_i(\omega)}$
$t_i(\omega)$

Definition For each individual $i \in I$ there is a *type mapping* $t_i : \Omega \to \bigcup_{\alpha \in \mathcal{A}} \Delta(S_\alpha)$, which is measurable in the sense that for every $S \in \mathcal{S}$ and $Q \in \mathcal{F}_{\Delta(S)}$ we have $t_i^{-1}(Q) \cap S \in \mathcal{F}_S$, for all $S \in \mathcal{S}$.

We require the type mapping t_i to satisfy the following properties:

(0) *Confinement:* If $\omega \in S'$ then $t_i(\omega) \in \Delta(S)$ for some $S \preceq S'$.

(1) If $S'' \succeq S' \succeq S$, $\omega \in S''$, and $t_i(\omega) \in \triangle(S)$ then $t_i(\omega_{S'}) = t_i(\omega)$.

(2) If $S'' \succeq S' \succeq S$, $\omega \in S''$, and $t_i(\omega) \in \triangle(S')$ then $t_i(\omega_S) = t_i(\omega)_{|S}$.

(3) If $S'' \succeq S' \succeq S$, $\omega \in S''$, and $t_i(\omega_{S'}) \in \triangle(S)$ then $S_{t_i(\omega)} \succeq S$.

Remark Property 1 is implied by the other properties.

Assumption (Introspection) If $\left\{\omega' \in \Omega : t_i(\omega')_{|S_{t_i(\omega)}} = t_i(\omega)\right\} \subseteq E$, E an event, then $t_i(\omega)(E) = 1$.

Definition For $i \in I$ and an event E, define the *awareness operator*

$$A_i\left(E\right) := \left\{\omega \in \Omega : t_i\left(\omega\right) \in \Delta\left(S\right), S \succeq S\left(E\right)\right\}$$

if there is a state ω such that $t_i(\omega) \in \Delta(S)$ with $S \succeq S(E)$, and by

$$A_i(E) := \emptyset^{S(E)}$$

otherwise.
The *unawareness operator* of individual $i \in I$ on events is now defined by

$$U_i(E) = \neg A_i(E).$$

Lemma If E is a (not necessarily measurable) event, then $A_i(E)$ (and thus $U_i(E)$) is an $S\left(E\right)$-based event.

Definition For $i \in I$, $p \in [0,1]$ and an event E, the *p-belief operator* is defined, as usual, by

$$B_i^p(E) := \{\omega \in \Omega : t_i(\omega)(E) \geq p\},$$

if there is a state ω such that $t_i(\omega)(E) \geq p$, and by

$$B_i^p(E) := \emptyset^{S(E)}$$

otherwise.

Proposition If E is an event, then $B_i^p(E)$ is an $S(E)$-based event.

Proposition (Standard Properties) Let E and F be events, $\{E_l\}_{l=1,2,\ldots}$ be an at most countable collection of events, and $p, q \in [0,1]$. The following properties of belief obtain:

(o) $B_i^p(E) \subseteq B_i^q(E)$, for $q \leq p$,

(i) Necessitation: $B_i^1(\Omega) = \Omega$,

(ii) Additivity: $B_i^p(E) \subseteq \neg B_i^q(\neg E)$, for $p + q > 1$,

(iiia) $B_i^p\left(\bigcap_{l=1}^{\infty} E_l\right) \subseteq \bigcap_{l=1}^{\infty} B_i^p(E_l)$,

(iiib) for any decreasing sequence of events $\{E_l\}_{l=1}^{\infty}$, $B_i^p\left(\bigcap_{l=1}^{\infty} E_l\right) = \bigcap_{l=1}^{\infty} B_i^p(E_l)$,

(iiic) $B_i^1\left(\bigcap_{l=1}^{\infty} E_l\right) = \bigcap_{l=1}^{\infty} B_i^1(E_l)$,

(iv) Monotonicity: $E \subseteq F$ implies $B_i^p(E) \subseteq B_i^p(F)$,

(v) Introspection: $B_i^p(E) \subseteq B_i^1 B_i^p(E)$.

Proposition Let E be an event and $p, q \in [0,1]$. The following properties of awareness and belief obtain:

1. Plausibility: $U_i(E) \subseteq \neg B_i^p(E) \cap \neg B_i^p \neg B_i^p(E)$,
2. Strong Plausibility: $U_i(E) \subseteq \bigcap_{n=1}^{\infty} \left(\neg B_i^p\right)^n (E)$,
3. B^pU Introspection: $B_i^p U_i(E) = \emptyset^{S(E)}$for $p \in (0,1]$, $B_i^0 U_i(E) = A_i(E)$,
4. AU Introspection: $U_i(E) = U_i U_i(E)$,
5. Weak Necessitation: $A_i(E) = B_i^1\left(S(E)^{\uparrow}\right)$,
6. $B_i^p(E) \subseteq A_i(E)$, $B_i^0(E) = A_i(E)$,
7. $B_i^p(E) \subseteq A_i B_i^q(E)$,
8. Symmetry: $A_i(E) = A_i(\neg E)$,
9. A Conjunction: $\bigcap_{\lambda \in L} A_i\left(E_\lambda\right) = A_i\left(\bigcap_{\lambda \in L} E_\lambda\right)$,
10. AB^p Self Reflection: $A_i B_i^p(E) = A_i(E)$,
11. AA Self Reflection: $A_i A_i(E) = A_i(E)$,
12. $B_i^p A_i(E) = A_i(E)$.

Belief	**Awareness**
mutual belief $B^p(E) := \bigcap_{i \in I} B_i^p(E)$	mutual awareness $A(E) := \bigcap_{i \in I} A_i(E)$
common certainty $CB^1\left(E\right) := \bigcap_{n=1}^{\infty}(B^1)^n(E)$	common awareness $CA(E) := \bigcap_{n=1}^{\infty}\left(A\right)^n\left(E\right)$

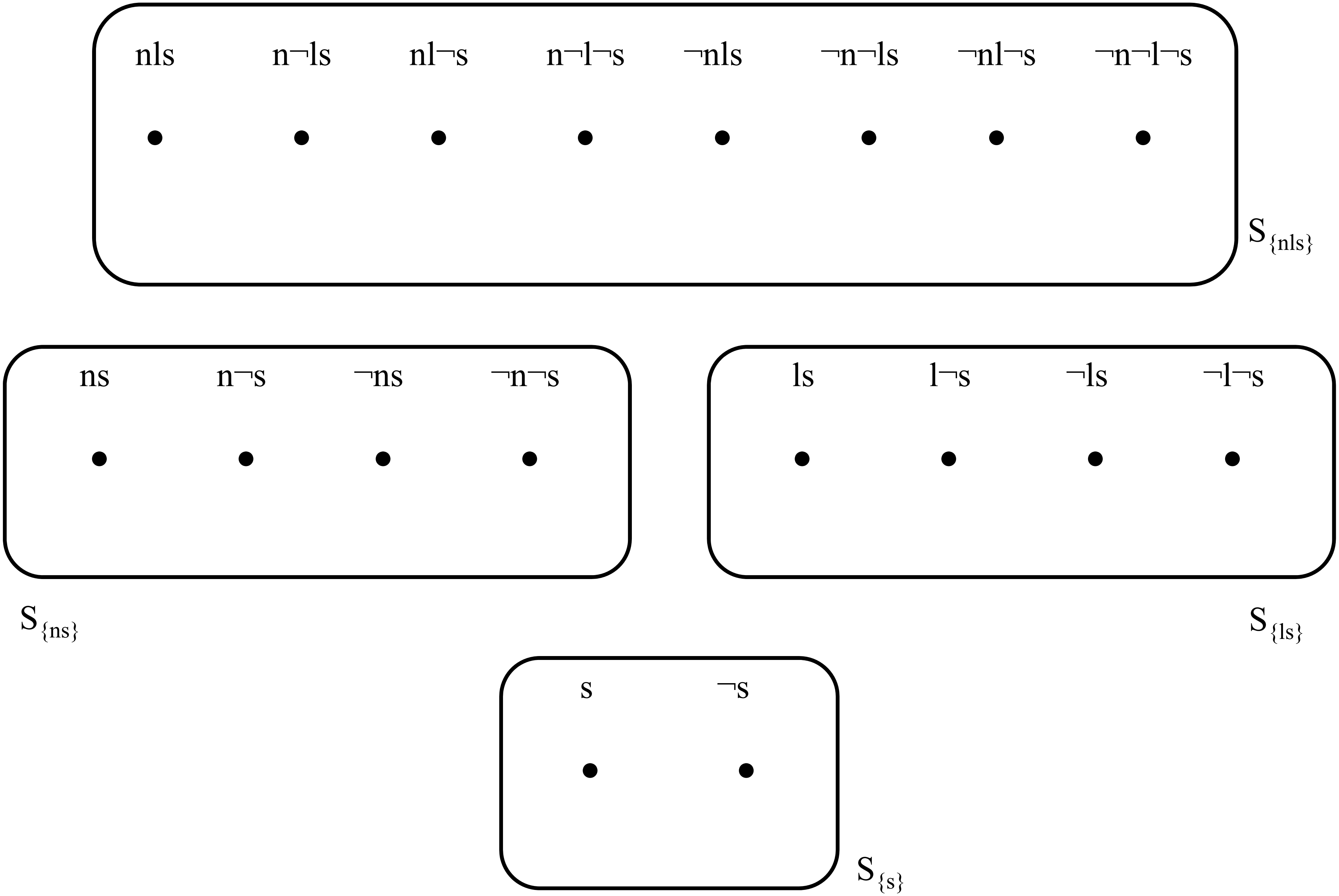
Proposition Let E be an event and $p, q \in [0,1]$. The following multi-person properties obtain:

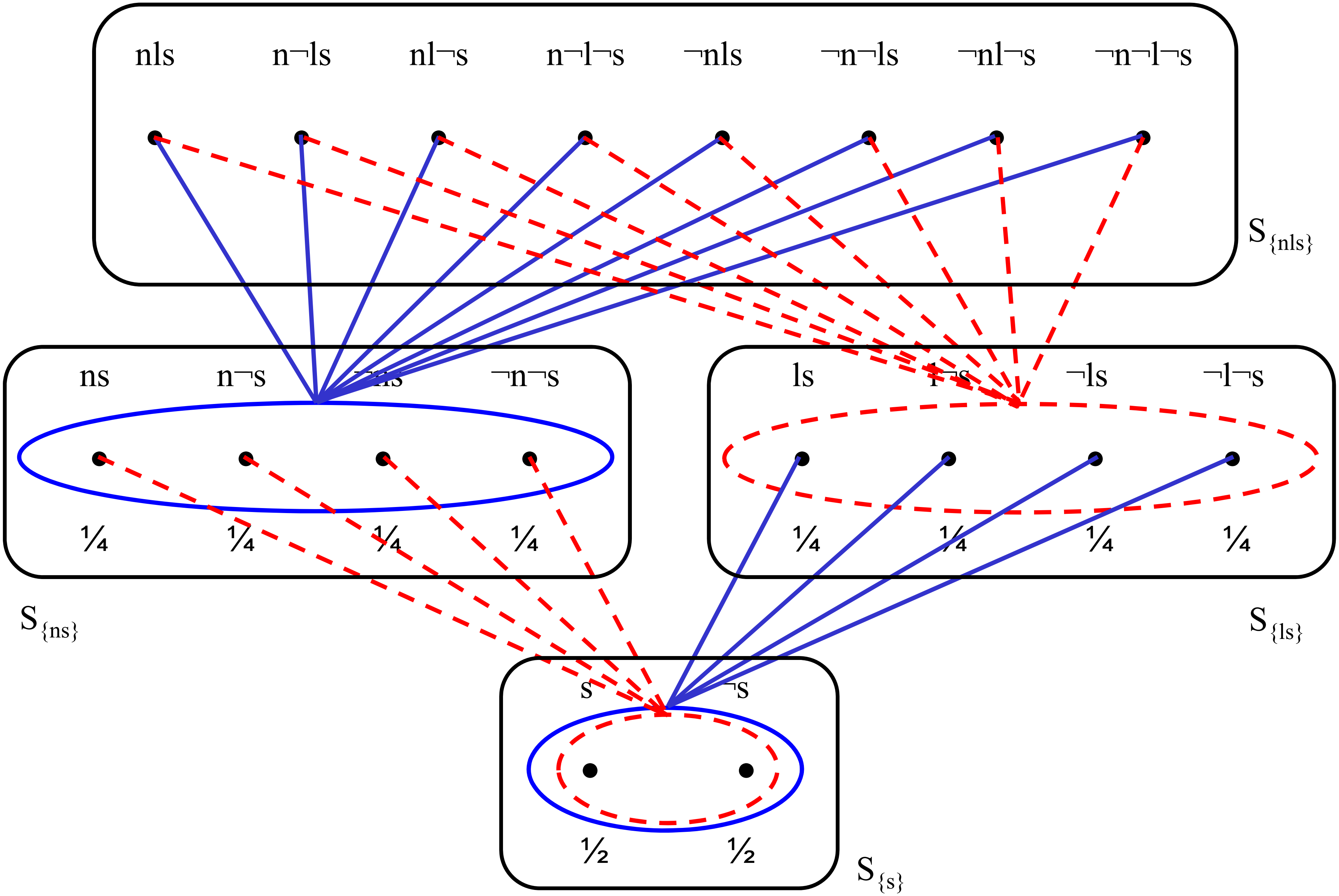
1. $A_i(E) = A_i A_j(E)$,
2. $A_i(E) = A_i B_j^p(E)$,
3. $B_i^p(E) \subseteq A_i B_j^q(E)$,
4. $B_i^p(E) \subseteq A_i A_j(E)$,
5. $CA(E) = A(E)$,
6. $CB^1(E) \subseteq CA(E)$,
7. $B^p(E) \subseteq A(E)$, $B^0(E) = A(E)$,
8. $B^p(E) \subseteq CA(E)$, $B^0(E) = CA(E)$,
9. $A(E) = B^1(S(E)^{\uparrow})$,
10. $CA(E) = B^1(S(E)^{\uparrow})$,
11. $CB^1(S(E)^{\uparrow}) \subseteq A(E)$,
12. $CB^1(S(E)^{\uparrow}) \subseteq CA(E)$.

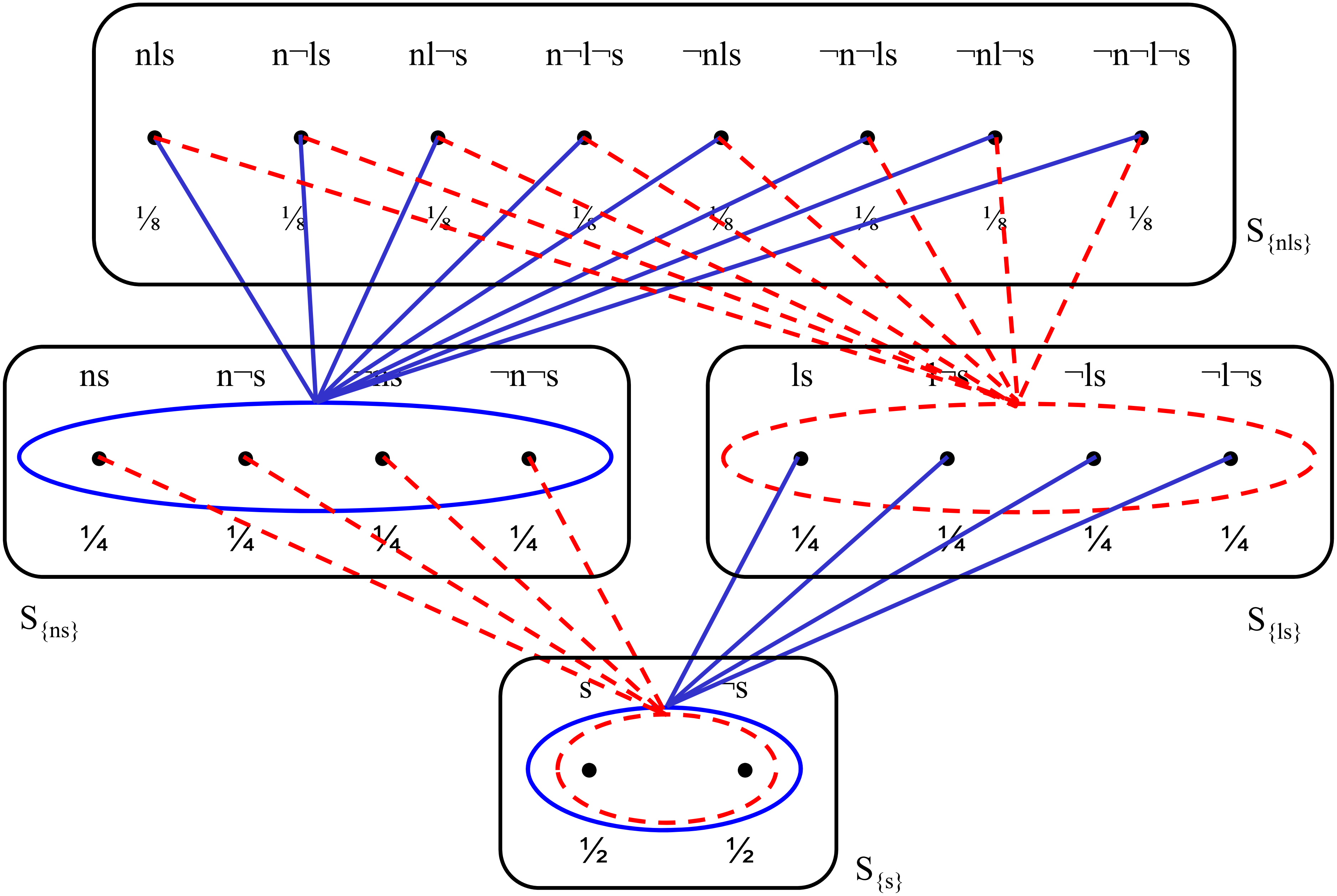
4. Speculation

Revisiting to the speculative trade example

- Status quo value of the firm depends on high ($100) or low ($80) sales, which is equiprobable.
- Both whether or not there is a lawsuit and whether or not there is a novelty are equiprobably too.
- Moreover, all events are independent.


nls
n¬ls
nl¬s
n¬l¬s
¬nls
¬n¬ls
¬nl¬s
¬n¬l¬s
$S_{\{nls\}}$
ns
n¬s
¬ns
¬n¬s
$S_{\{ns\}}$
ls
l¬s
¬ls
¬l¬s
$S_{\{ls\}}$
s
¬s
$S_{\{s\}}$


nls
n¬ls
nl¬s
n¬l¬s
¬nls
¬n¬ls
¬nl¬s
¬n¬l¬s
$S_{\{nls\}}$
ns
n¬s
¬ns
¬n¬s
¼
¼
¼
¼
$S_{\{ns\}}$
ls
l¬s
¬ls
¬l¬s
¼
¼
¼
¼
$S_{\{ls\}}$
s
¬s
½
½
$S_{\{s\}}$


nls
n¬ls
nl¬s
n¬l¬s
¬nls
¬n¬ls
¬nl¬s
¬n¬l¬s
⅛
⅛
⅛
⅛
⅛
⅛
⅛
⅛
S_{nls}
ns
n¬s
¬ns
¬n¬s
¼
¼
¼
¼
S_{ns}
ls
l¬s
¬ls
¬l¬s
¼
¼
¼
¼
S_{ls}
s
¬s
½
½
S_{s}

Revisiting to the speculative trade example

- There is a common prior (i.e., a projective system of probability measures with which agents' types are consistent) and common certainty of willingness to trade but each agent has a strict preference to trade.
- Counterexample to Milgrom and Stokey (1982).
- Speculative trade is possible under unawareness.
- Knife-edge case???

Prior in a standard type space S (Samet, 1998):

For every event $E \in \mathcal{F}_S$, $P_i^S(E) = \int_S t_i(\cdot)(E)\, dP_i^S(\cdot)$.

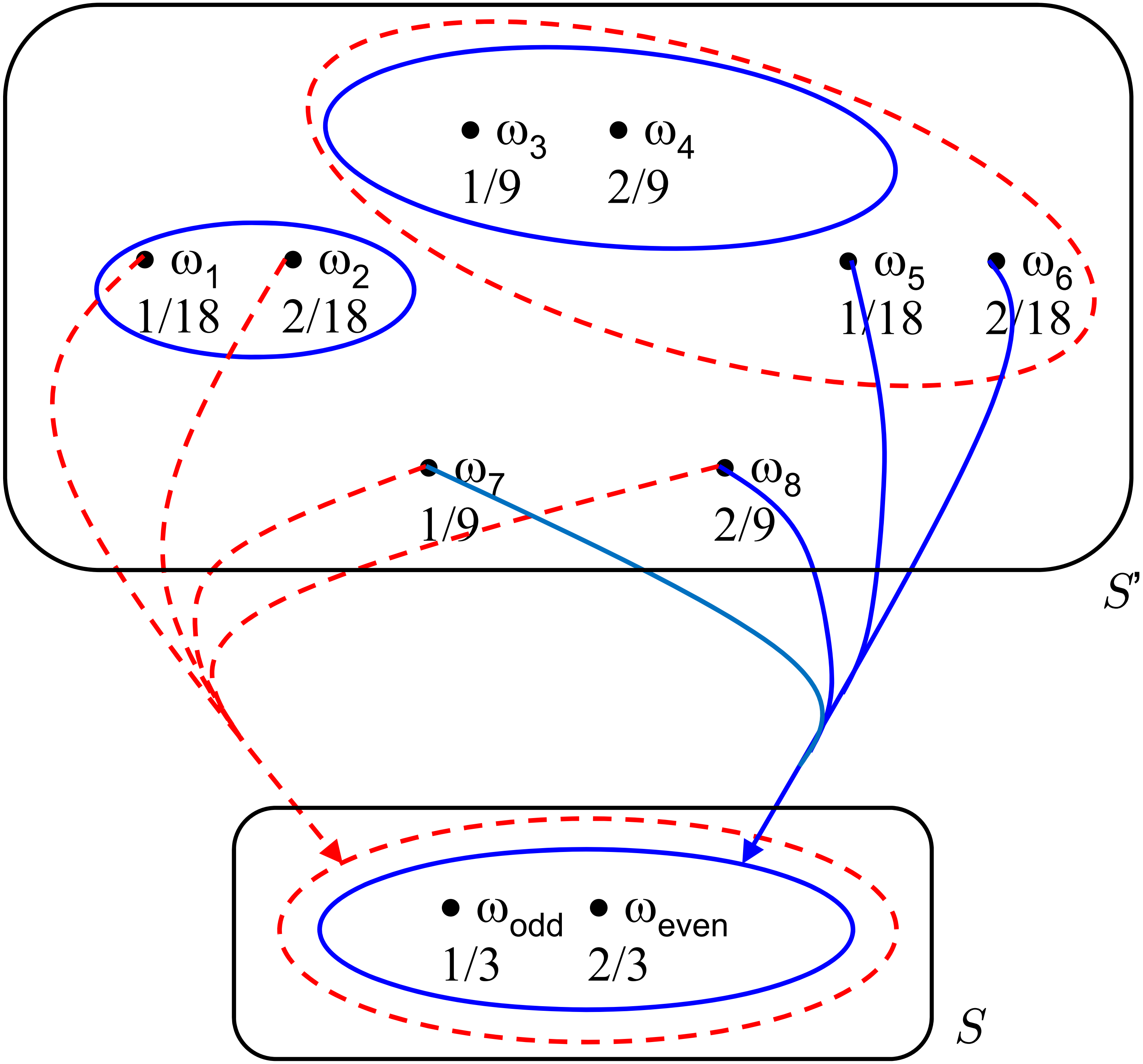
- convex combination of posteriors (types)
- a consistency condition on types instead of a belief at a "prior stage"

Definition (Prior) A prior for player i is a system of probability measures $P_i = \left(P_i^S\right)_{S\in\mathcal{S}} \in \prod_{S\in\mathcal{S}} \Delta(S)$ such that

1. The system is projective: If $S' \preceq S$ then the marginal of P_i^S on S' is $P_i^{S'}$. (That is, if $E \in \Sigma$ is an event whose base-space $S(E)$ is lower or equal to S', then $P_i^S(E) = P_i^{S'}(E)$.)
2. Each probability measure P_i^S is a convex combination of i's beliefs in S: For every event $E \in \Sigma$ such that $S(E) \preceq S$,

$$P_i^S\left(E \cap S \cap A_i(E)\right) = \int_{S\cap A_i(E)} t_i(\cdot)(E)\, dP_i^S(\cdot).$$

$P = \left(P^S\right)_{S\in\mathcal{S}} \in \prod_{S\in\mathcal{S}} \Delta(S)$ is a common prior if P is a prior for every player $i \in I$.



ω3
1/9
ω4
2/9
ω1
1/18
ω2
2/18
ω5
1/18
ω6
2/18
ω7
1/9
ω8
2/9
S'
ωodd
1/3
ωeven
2/3
S

$[t_i(\omega)] := \{\omega' \in \Omega : t_i(\omega') = t_i(\omega)\}$

Definition (Prior) A common prior $P = \left(P^S\right)_{S \in \mathcal{S}} \in \prod_{S \in \mathcal{S}} \Delta(S)$ is *positive* if and only if for all $i \in I$ and $\omega \in \Omega$: If $t_i\left(\omega\right) \in \triangle\left(S'\right)$, then $[t_i(\omega)] \cap S' \in \mathcal{F}_{S'}$ and $P^S\left(\left([t_i\left(\omega\right)] \cap S'\right)^{\uparrow} \cap S\right) > 0$ for all $S \succeq S'$.

Let x_1 and x_2 be real numbers and v a random variable on Ω. Define the sets $E_1^{\leq x_1} := \left\{\omega \in \Omega : \int_{S_{t_1(\omega)}} v\left(\cdot\right) d\left(t_1\left(\omega\right)\right)\left(\cdot\right) \leq x_1\right\}$ and $E_2^{\geq x_2} := \left\{\omega \in \Omega : \int_{S_{t_2(\omega)}} v\left(\cdot\right) d\left(t_2\left(\omega\right)\right)\left(\cdot\right) \geq x_2\right\}$.
We say that at ω, conditional on his information, player 1 (resp. player 2) believes that the expectation of v is weakly below x_1 (resp. weakly above x_2) if and only if $\omega \in E_1^{\leq x_1}$ (resp. $\omega \in E_2^{\geq x_2}$).

Theorem Consider a finite unawareness belief structure with a positive common prior $P = \left(P^S\right)_{S \in \mathcal{S}} \in \prod_{S \in \mathcal{S}} \Delta(S)$. Then there is no state $\tilde{\omega} \in \Omega$ such that there are a random variable $v : \Omega \longrightarrow \mathbb{R}$ and $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$, with the following property: at $\tilde{\omega}$ it is common certainty that conditional on her information, player 1 believes that the expectation of v is weakly below x_1 and, conditional on his information, player 2 believes that the expectation of v is weakly above x_2.

Arbitrary small transaction costs (like a Tobin tax) rule out speculative trade under unawareness.

Converse to the theorem is false.

5. Bayesian games with unawareness

Bayesian games with unawareness

- Unawareness of (payoff-)relevant events, actions, and players.
- Replace type space with unawareness type space.
- Digression: How do we model uncertainty over action sets in standard games?

Definition A Bayesian game with unawareness

$$\Gamma(\mathcal{S}) = \left\langle \langle \mathcal{S}, \preceq \rangle, \left(r_{S_\beta}^{S_\alpha} \right)_{S_\beta \preceq S_\alpha}, (t_i)_{i \in I}, (M_i)_{i \in I}, (\mathcal{M}_i)_{i \in I}, (u_i)_{i \in I} \right\rangle$$

consists of a finite unawareness type space
$\left\langle \langle \mathcal{S}, \preceq \rangle, \left(r_{S_\beta}^{S_\alpha} \right)_{S_\beta \preceq S_\alpha}, (t_i)_{i \in I} \right\rangle$ and

(i) a nonempty finite set of actions M_i, for $i \in I$, and a correspondence $\mathcal{M}_i : \Omega \longrightarrow 2^{M_i} \setminus \{\emptyset\}$, for $i \in I$, such that for any nonempty subset of actions[1] $M_i' \subseteq M_i$, $[M_i'] := \{\omega \in \Omega : M_i' \subseteq \mathcal{M}_i(\omega)\}$ is an event (in the unawareness type space), and $\omega', \omega'' \in [t_i(\omega)] \cap S_{t_i(\omega)}$ implies $\mathcal{M}_i(\omega') = \mathcal{M}_i(\omega'')$, for all $\omega \in \Omega$,

(ii) for every $i \in I$, a utility function $u_i : \bigcup_{\omega \in \Omega} \left(\left(\prod_{j \in I} \mathcal{M}_j(\omega) \right) \times \{\omega\} \right) \longrightarrow \mathbb{R}$.

A *strategy* of player i in a Bayesian game with unawareness is a function $\sigma_i : \Omega \longrightarrow \Delta(M_i)$ such that for all $\omega \in \Omega$,

(i) $\sigma_i(\omega) \in \Delta\left(\mathcal{M}_i(\omega_{S_{t_i(\omega)}})\right)$, and

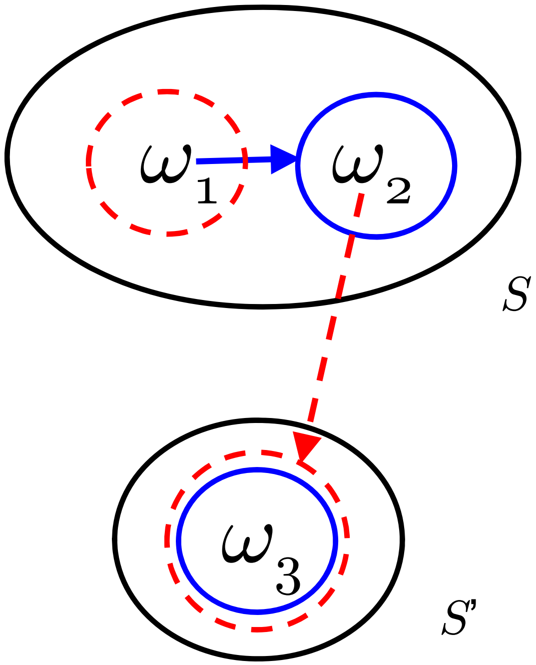
(ii) $t_i(\omega') = t_i(\omega)$ implies $\sigma_i(\omega') = \sigma_i(\omega)$.

Example (Feinberg 2005)

Colin

		b_1	b_2	b_3
Rowena	a_1	0, 2	3, 3	0, 2
	a_2	$\boxed{2, 2}$	2, 1	2, 1
	a_3	1, 0	4, 0	0, 1

Unique Nash equilibrium (a_2, b_1)



		Colin		
		b_1	b_2	b_3
Rowena	a_1	$0, 2$	$3, 3$	$0, 2$
	a_2	$2, 2$	$2, 1$	$2, 1$
	a_3	$1, 0$	$4, 0$	$0, 1$

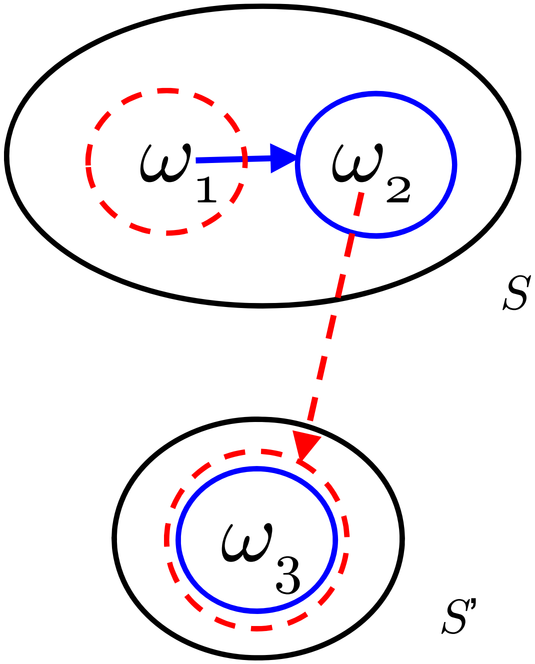
		Colin		
		b_1	b_2	b_3
Rowena	a_1	$0, 2$	$\boxed{3, 3}$	$0, 2$
	a_2	$\boxed{2, 2}$	$2, 1$	$2, 1$

$$\mathcal{M}_R(\omega_1) = \mathcal{M}_R(\omega_2) = \{a_1, a_2, a_3\}, \mathcal{M}_R(\omega_3) = \{a_1, a_2\},$$

$$\mathcal{M}_C(\omega_1) = \mathcal{M}_C(\omega_2) = \mathcal{M}_C(\omega_3) = \{b_1, b_2, b_3\}.$$

An equilibrium

$$(\sigma_R(\omega), \sigma_C(\omega)) = \begin{cases} (a_3, b_3) & \text{if} \quad \omega = \omega_1 \\ (a_3, b_2) & \text{if} \quad \omega = \omega_2 \\ (a_1, b_2) & \text{if} \quad \omega = \omega_3 \end{cases}$$



Colin

Rowena

	b_1	b_2	b_3
a_1	0, 2	3, 3	0, 2
a_2	2, 2	2, 1	2, 1
a_3	1, 0	4, 0	0, 1

Colin

Rowena

	b_1	b_2	b_3
a_1	0, 2	$\boxed{3, 3}$	0, 2
a_2	$\boxed{2, 2}$	2, 1	2, 1

$$\mathcal{M}_R(\omega_1) = \mathcal{M}_R(\omega_2) = \{a_1, a_2, a_3\}, \mathcal{M}_R(\omega_3) = \{a_1, a_2\},$$

$$\mathcal{M}_C(\omega_1) = \mathcal{M}_C(\omega_2) = \mathcal{M}_C(\omega_3) = \{b_1, b_2, b_3\}.$$

An equilibrium

$$(\sigma_R(\omega), \sigma_C(\omega)) = \begin{cases} (a_3, b_3) & \text{if} \quad \omega = \omega_1 \\ (a_3, b_2) & \text{if} \quad \omega = \omega_2 \\ (a_1, b_2) & \text{if} \quad \omega = \omega_3 \end{cases}$$

Denote $\sigma_{S_{t_i(\omega)}} := \left((\sigma_j(\omega'))_{j\in I}\right)_{\omega'\in S_{t_i(\omega)}}$. The *expected utility* of player-type $(i, t_i(\omega))$ from the strategy profile $\sigma_{S_{t_i(\omega)}}$ is given by

$$U_{(i,t_i(\omega))}(\sigma_{S_{t_i(\omega)}}) :=$$

$$\sum_{\omega'\in S_{t_i(\omega)}} \sum_{m\in\prod_{j\in I}\mathcal{M}_j\left(\omega'_{S_{t_j(\omega')}}\right)} \left(\prod_{j\in I}\sigma_j(\omega')\left(\{m_j\}\right)\right)\cdot u_i\left((m_j)_{j\in I},\omega'\right)t_i(\omega)(\{\omega'\}).$$

Definition Given a Bayesian game with unawareness $\Gamma(\mathcal{S})$, define the associated strategic game by

(i) $\{(i, t_i(\omega)) : \omega \in \Omega$ and $i \in I\}$ is the set of players,

and for each player $(i, t_i(\omega))$,

(ii) the set of mixed strategies is $\Delta(\mathcal{M}_i(\omega_{S_{t_i(\omega)}})$, and

(iii) the utility function is given above.

A profile $(\sigma_i)_{i \in I}$ is an equilibrium of the Bayesian game with unawareness if and only if the following is an equilibrium of the associated strategic game: $(i, t_i(\omega))$ plays $\sigma_i(\omega)$, for all $i \in I$ and $\omega \in \Omega$.

Proposition (Existence) Every finite Bayesian game with unawareness has an equilibrium.

Sublattice with least upper bound S is $l(S) := \{S' \in \mathcal{S} : S' \preceq S\}$

Given $\Gamma(\mathcal{S})$, the S-partial game is $\Gamma(l(S))$.

At $\omega \in \Omega$, player i views the game as given by $\Gamma(l(S_{t_i(\omega)}))$.

Proposition Given a Bayesian game with unawareness $\Gamma(\mathcal{S})$, consider for $S', S'' \in \mathcal{S}$ with $S' \preceq S''$ the S'-partial (resp. S'' -partial) Bayesian game with unawareness. If I, Ω, and $(M_i)_{i \in I}$ are finite, then for every equilibrium of the S'-partial Bayesian game, there is an equilibrium of the S''-partial Bayesian game in which equilibrium strategies of player-types in $\{(i, t_i(\omega)) : \omega \in \Omega' = \bigcup_{S \in l(S')} S \text{ and } i \in I\}$ are identical with the equilibrium strategies in the S'-partial Bayesian game.

Remark Consider for $S', S'' \in \mathcal{S}$ with $S' \preceq S''$ the S'-partial (resp. S''-partial) Bayesian game with unawareness of given game $\Gamma(\mathcal{S})$. Then for every equilibrium of the S''-partial Bayesian game there is a unique equilibrium of the S'-partial Bayesian game in which the equilibrium strategies of player-types in $\{(i, t_i(\omega)) : \omega \in \Omega' = \bigcup_{S \in l(S')} S \text{ and } i \in I\}$ are identical to the equilibrium strategies of the S''-partial Bayesian game.

(Peleg and Tijs, 1996, Peleg, Potters, and Tijs, 1996)

Unawareness perfection

γ^0	L	R
U	$1,1$	$2,0$
D	$0,2$	$2,2$

Unawareness perfection

γ^0	L	R
U	$1,1$	$2,0$
D	$0,2$	$2,2$

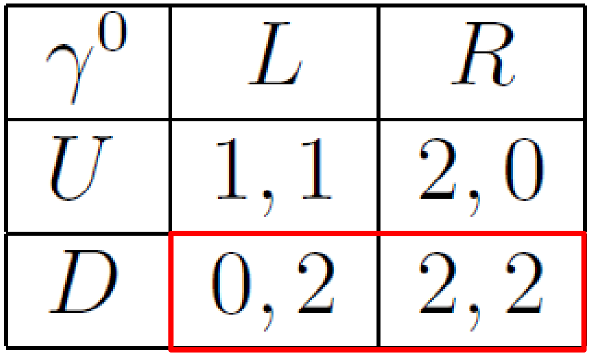
Unawareness perfection

γ^0	L	R
U	$1, 1$	$2, 0$
D	$0, 2$	$2, 2$

Unawareness perfection

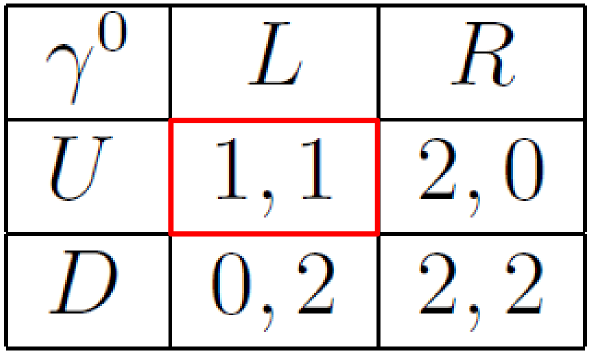
γ^0	L	R
U	$1, 1$	$2, 0$
D	$0, 2$	$2, 2$

Unawareness perfection



γ^0	L	R
U	$1, 1$	$2, 0$
D	$0, 2$	$2, 2$

Unawareness perfection

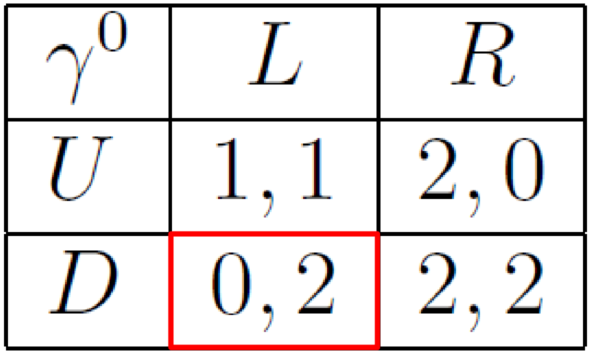


γ^0	L	R
U	$1, 1$	$2, 0$
D	$0, 2$	$2, 2$

Unawareness perfection

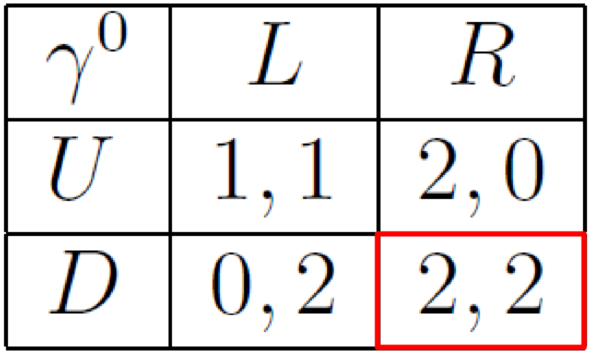
γ^0	L	R
U	$1, 1$	$2, 0$
D	$0, 2$	$2, 2$

Unawareness perfection



γ^0	L	R
U	1, 1	2, 0
D	$\boxed{0, 2}$	2, 2

Unawareness perfection



γ^0	L	R
U	$1, 1$	$2, 0$
D	$0, 2$	$2, 2$

γ^0	L	R
U	$1, 1$	$2, 0$
D	$0, 2$	$2, 2$

γ^1	L
U	$1, 1$
D	$0, 2$

γ^2	R
U	$2, 0$
D	$2, 2$

γ^3	L	R
U	$1, 1$	$2, 0$

γ^4	L	R
D	$0, 2$	$2, 2$

γ^5	L
U	$1, 1$

γ^6	L
D	$0, 2$

γ^7	R
U	$2, 0$

γ^8	R
D	$2, 2$

- Opponents could have any awareness level
- Full support belief over types of opponent
- Consider limit belief over opponent’s types close to him being fully aware.
- Limit Bayesian Nash equilibrium selects (U, L), the undominated Nash equilibrium.

For each strategic game, construct a Bayesian game with unawareness in which each player has a full support belief over opponents' having any awareness of action sets.

Each such Bayesian game with unawareness has a Bayesian Nash equilibrium.

Definition An unawareness perfect equilibrium of a strategic game is a Nash equilibrium for which there exists a sequence Bayesian games with certainty of awareness at the limit and a corresponding sequence of Bayesian Nash equilibria such that the limit of the sequence at the true state converges to the Nash equilibrium.

Theorem For every finite strategic game, an unawareness perfect equilibrium exist.

Theorem A Nash equilibrium is an unawareness perfect equilibrium if and only if undominated (i.e., not weakly dominated).

Interpretation

Undominated Nash equilibrium: No opponents' action may be excluded from consideration.

Unawareness perfect equilibrium: Slight chance that any opponents' subset of action profiles may be excluded from consideration.

(Increases the probability that other not excluded actions are played.)

Comparison to *Trembling Hand Perfection*:

- Independence of opponents' mistakes.
- Mutual belief in opponents' mistakes.
- No assumption of irrational behavior / mistakes under unawareness perfection. Instead beliefs are formed over opponents' perception of the game.

6. Revealed Unawareness

Revealed Unawareness

- What’s the difference between a zero-probability event and unawareness?
- Can we reveal unawareness?

Savage confined subjective expected utility theory (SEU) to the decision maker’s small world.

The decision maker chooses among acts, i.e., mappings from states to outcomes.

Can the decision maker choose among acts without being made aware of events by the description of acts?

- Yes, in reality we do it all the times.
- No, in standard SEU models.

Proposal for resolution:

- Replace the state space in Savage by lattice of spaces.
- Define acts on the union of spaces.
- Let preferences depend on the space (i.e., the awareness level).
- The decision maker chooses among actions that are "names" for acts that are not necessarily understood by the decision maker. His understanding depends on her awareness level.

$$\begin{array}{ccccc}
\boxed{\begin{array}{cc} l & \neg\, l \\ 80 & 100 \end{array}} & & \boxed{\begin{array}{cc} l & \neg\, l \\ 100 & 100 \end{array}} & & \boxed{\begin{array}{cc} l & \neg\, l \\ 100 & 80 \end{array}} \\
 & \sim & & \sim & \\
\boxed{\begin{array}{c} \emptyset \\ 100 \end{array}} & & \boxed{\begin{array}{c} \emptyset \\ 100 \end{array}} & & \boxed{\begin{array}{c} \emptyset \\ 100 \end{array}} \\
\text{Contract 1} & & \text{Contract 2} & & \text{Contract 3}
\end{array}$$

Choice behavior is inconsistent with either “lawsuit” or “not lawsuit” being a Savage null event.

But it is consistent with unawareness of “lawsuit”.

Outline

1. Informal introduction
2. Epistemic models of unawareness
3. Type spaces with unawareness
4. Speculation
5. Bayesian games with unawareness
6. Revealed unawareness
7. Dynamic games with unawareness
8. …(???)

7. Dynamic games with unawareness

- How to model dynamic interactions between players with asymmetric awareness?

What are the problems:

- Players may be aware of different subsets players, moves of nature, or actions (i.e., different “representations” of the game)
- Players may become aware of further players, moves of nature or actions during the play of the game (i.e., endogenous changes of their “representations” of the game)

- What is a sensible solution concept?

  What are the problems:

  - Players couldn't learn equilibrium through repeated play (because of "novel" situations)
  - Subgame perfection, sequential equilibrium etc. ignore surprises but with unawareness players could be surprised frequently
  - Players may make others strategically aware. Thus players should be able to reason carefully when being surprised by others rather than discarding surprising actions of others as a mistake.

Introductory Example

- A glimpse of how we model unawareness in dynamic games
- what solution concept we suggest to these models
- Relevance: Show that outcomes under unawareness of an action may differ from outcomes under the unavailability of an action

Example: Battle of the Sexes

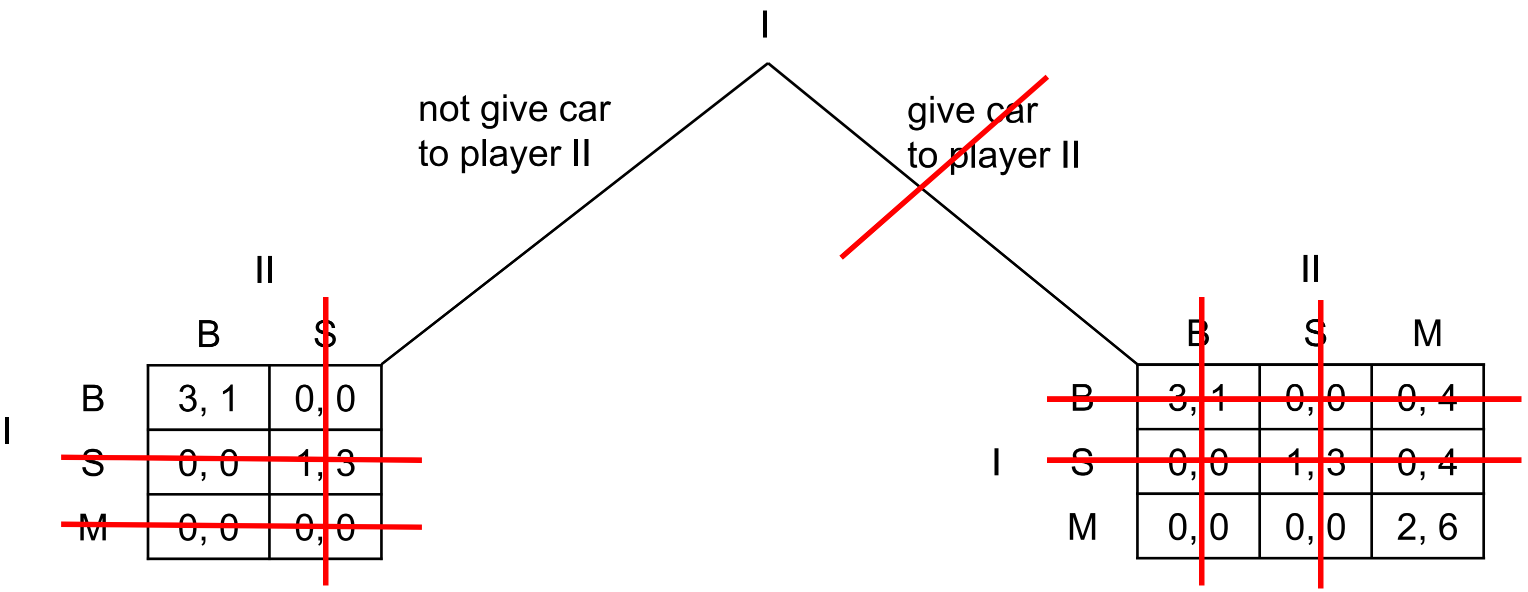
		II	
		B	S
I	B	3, 1	0, 0
	S	0, 0	1, 3

Example: Battle of the Sexes

II

		B	S	M
	B	3, 1	0, 0	0, 4
I	S	0, 0	1, 3	0, 4
	M	0, 0	0, 0	2, 6

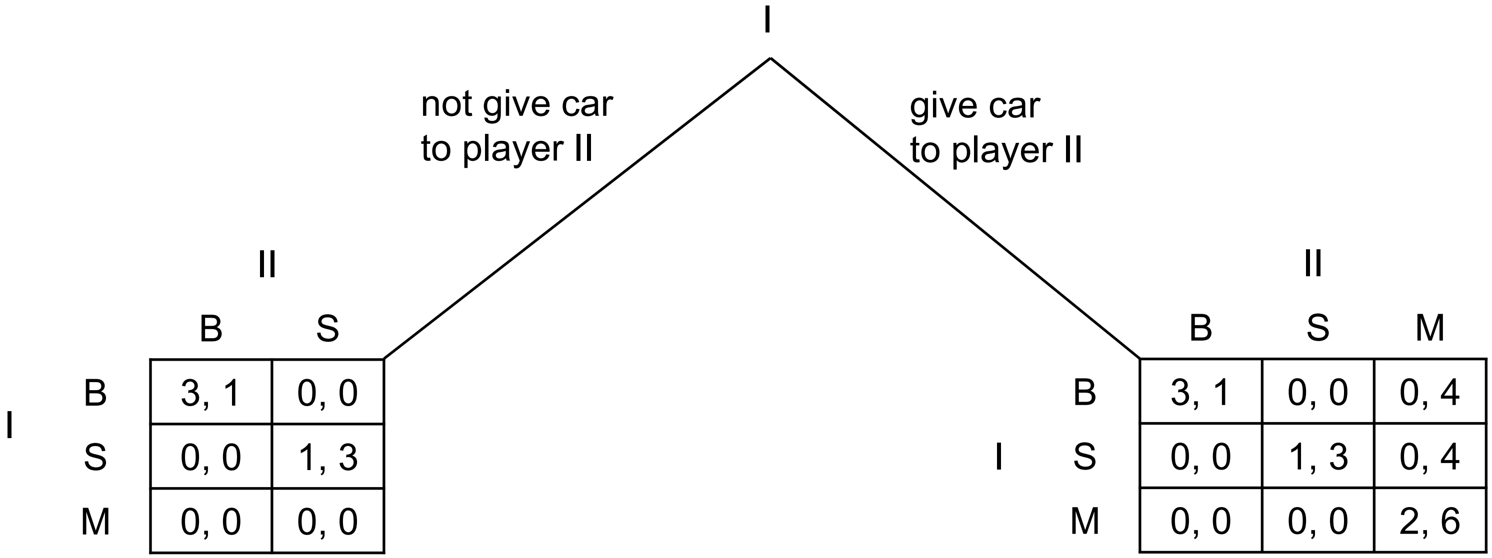
Example: A kind of “Battle of the Sexes”



S^1_{I} = {any strategy except ones with “M” after not giving the car}
S^1_{II}= {(B, M), (S, M)}
S^2_{I} = {(not give, B, M), (give, B, M)}
S^2_{II}= {(B, M), (S, M)}
S^3_{I} = {not give, B, M), (give, B, M)}
S^3_{II}= {(B, M)}
S^4_{I} = {not give, B, M)}
S^4_{II}= {(B, M)}

Unique extensive-form rationalizable outcome

Example: A kind of "Battle of the Sexes"



I

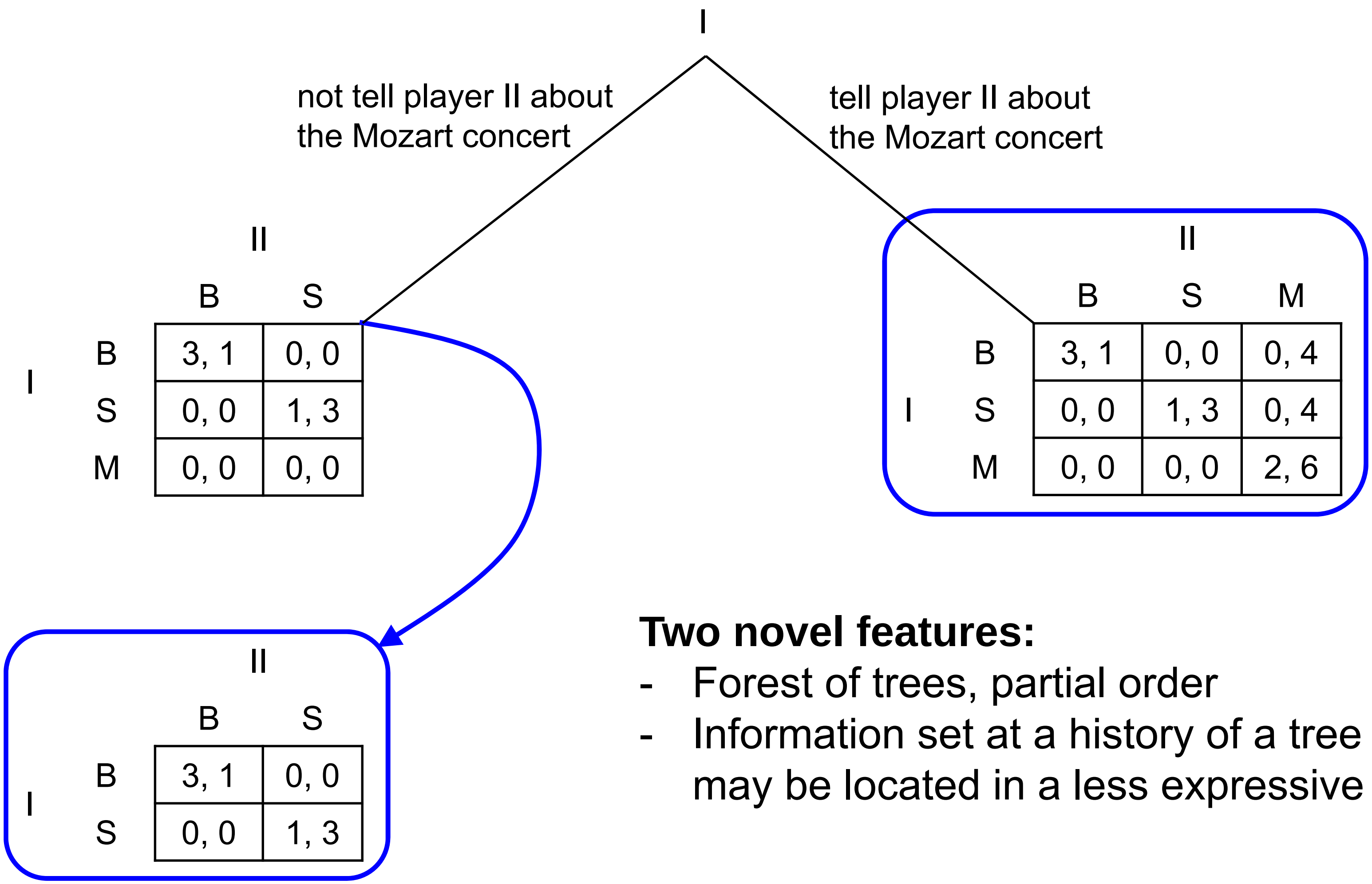
not give car
to player II

give car
to player II

		II	
		B	S
I	B	3, 1	0, 0
	S	0, 0	1, 3
	M	0, 0	0, 0

			II	
		B	S	M
	B	3, 1	0, 0	0, 4
I	S	0, 0	1, 3	0, 4
	M	0, 0	0, 0	2, 6

Example: A kind of “Battle of the Sexes”



II

I	B	S
B	3, 1	0, 0
S	0, 0	1, 3
M	0, 0	0, 0

II

I	B	S	M
B	3, 1	0, 0	0, 4
S	0, 0	1, 3	0, 4
M	0, 0	0, 0	2, 6

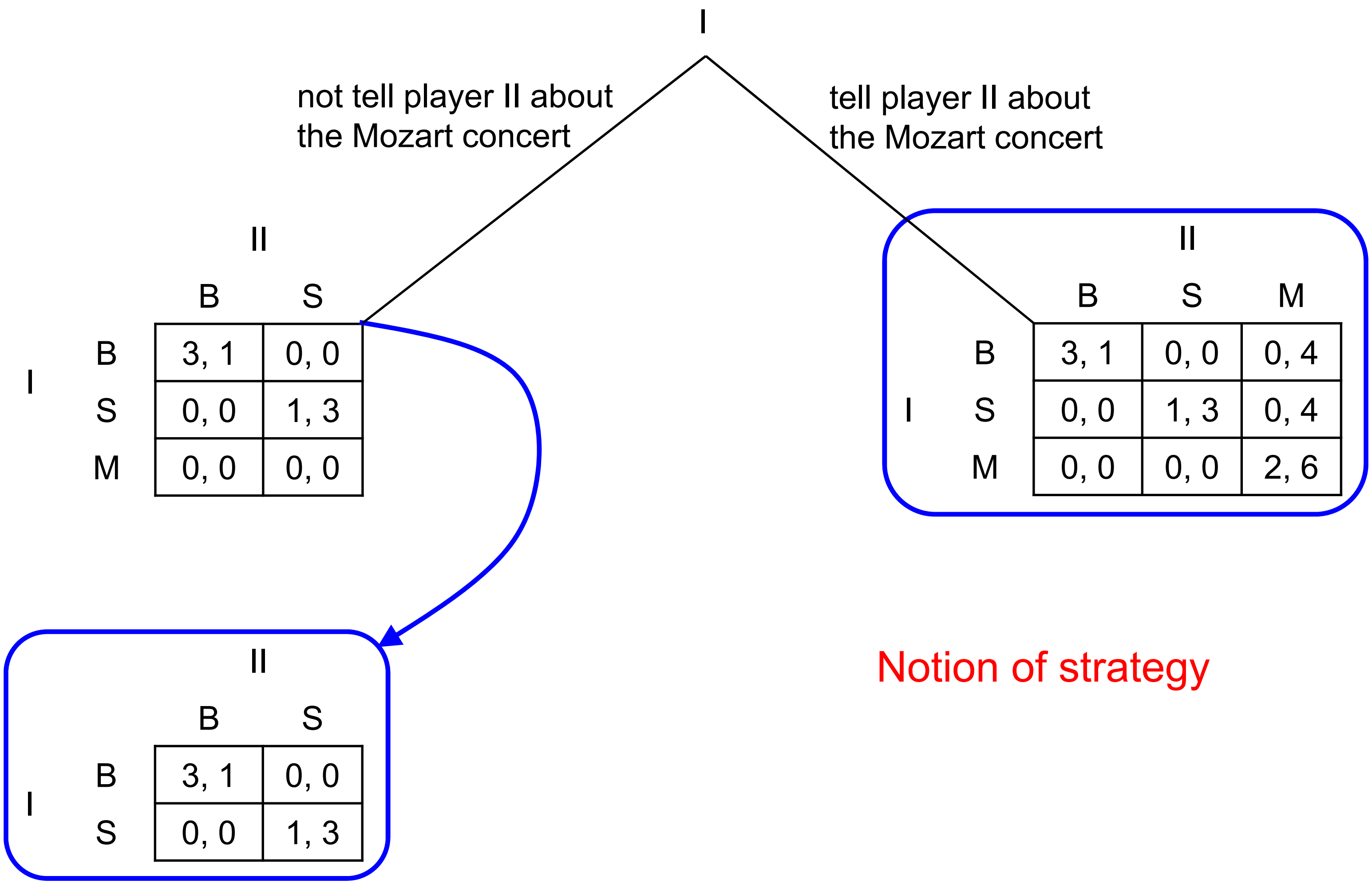
II

I	B	S
B	3, 1	0, 0
S	0, 0	1, 3

Two novel features:

- Forest of trees, partial order
- Information set at a history of a tree may be located in a less expressive tree

Example: A kind of “Battle of the Sexes”



not tell player II about the Mozart concert

tell player II about the Mozart concert

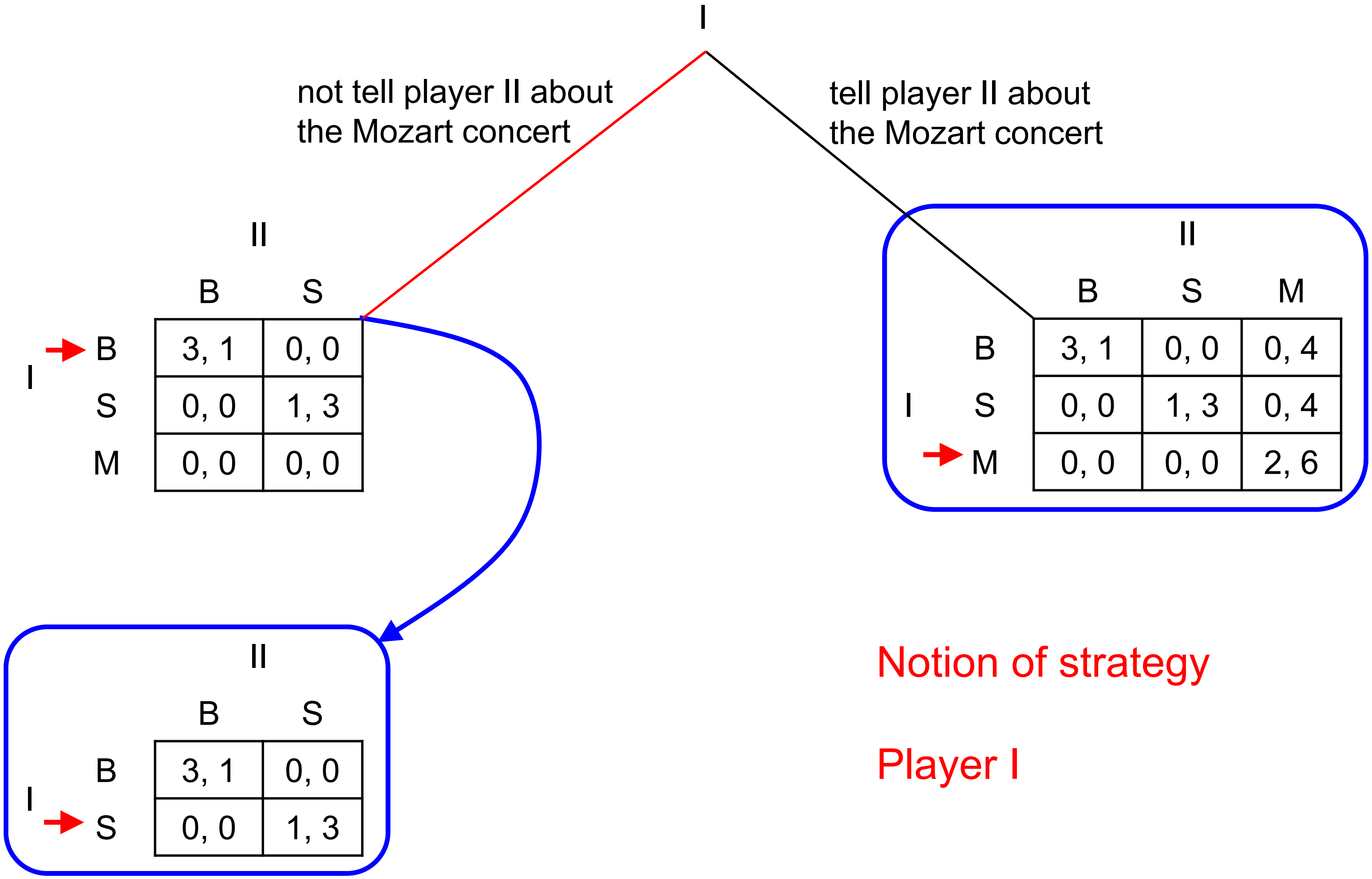
		II	
		B	S
I	B	3, 1	0, 0
	S	0, 0	1, 3
	M	0, 0	0, 0

		II		
		B	S	M
I	B	3, 1	0, 0	0, 4
	S	0, 0	1, 3	0, 4
	M	0, 0	0, 0	2, 6

		II	
		B	S
I	B	3, 1	0, 0
	S	0, 0	1, 3

Notion of strategy

Example: A kind of “Battle of the Sexes”



I

not tell player II about
the Mozart concert

tell player II about
the Mozart concert

II

I

	B	S
→ B	3, 1	0, 0
S	0, 0	1, 3
M	0, 0	0, 0

II

I

	B	S	M
B	3, 1	0, 0	0, 4
S	0, 0	1, 3	0, 4
→ M	0, 0	0, 0	2, 6

II

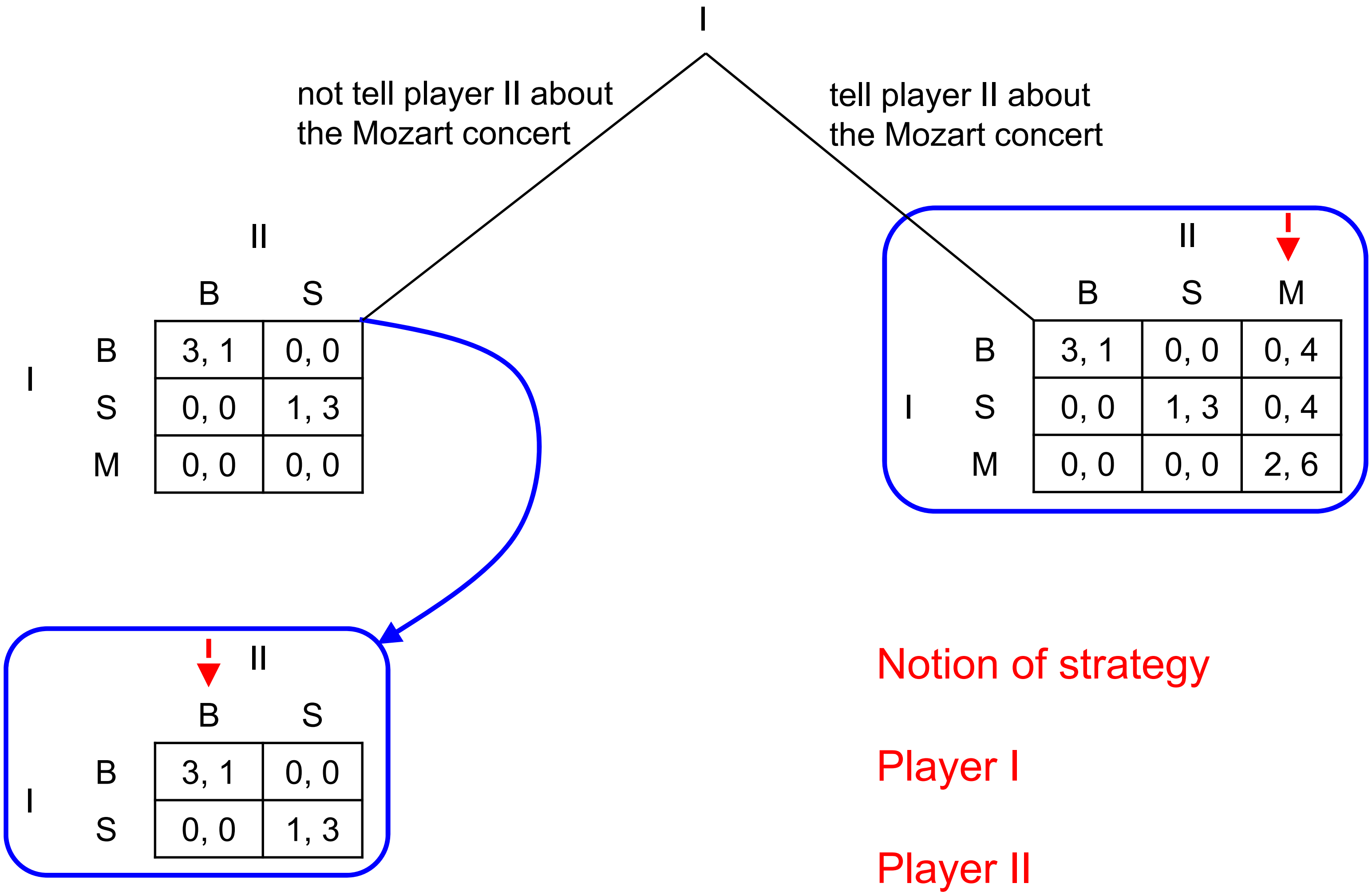
I

	B	S
B	3, 1	0, 0
→ S	0, 0	1, 3

Notion of strategy

Player I

Example: A kind of “Battle of the Sexes”



		B	S
	B	3, 1	0, 0
I	S	0, 0	1, 3
	M	0, 0	0, 0

		B	S	M
	B	3, 1	0, 0	0, 4
I	S	0, 0	1, 3	0, 4
	M	0, 0	0, 0	2, 6

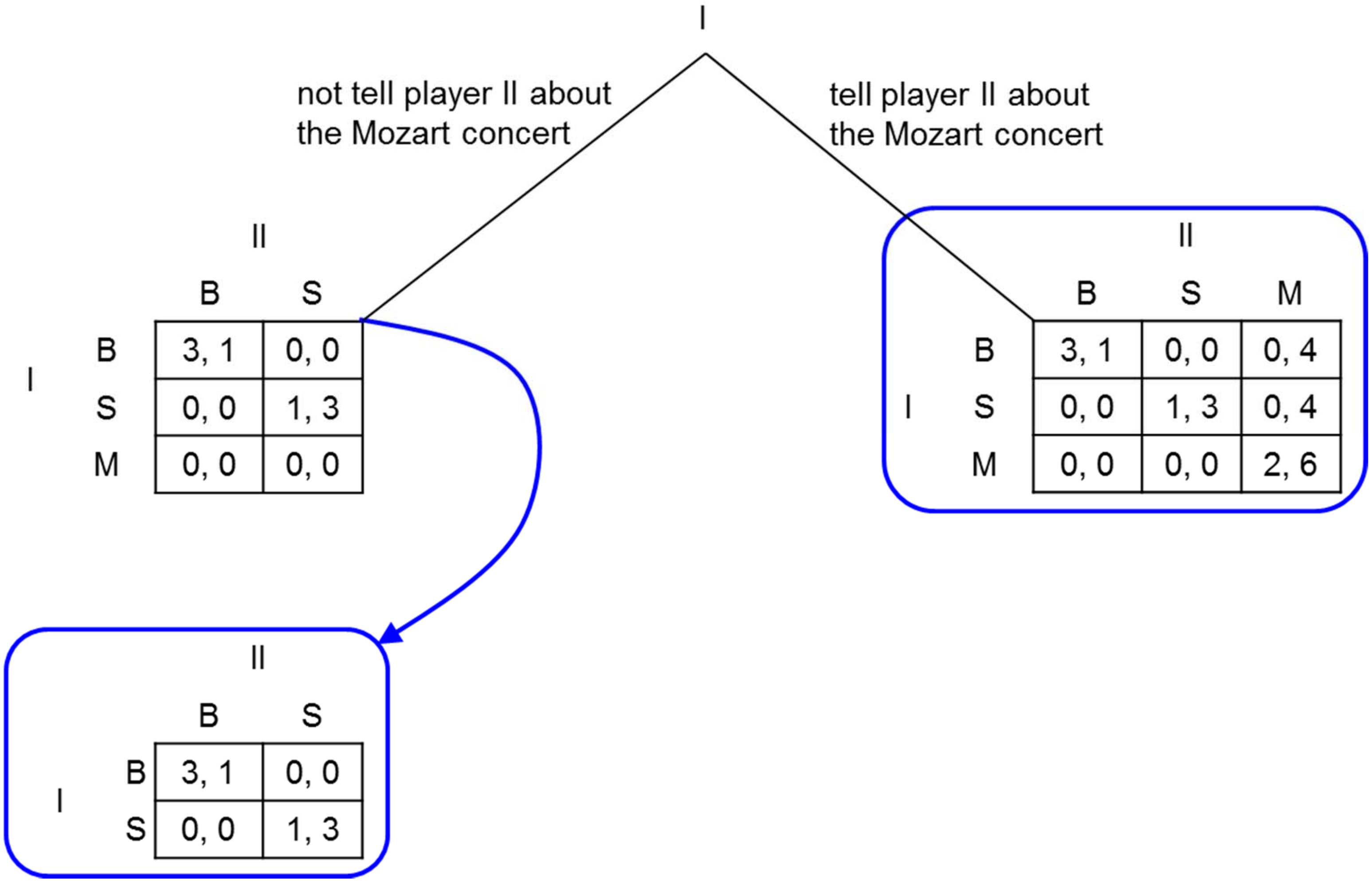
		B	S
I	B	3, 1	0, 0
	S	0, 0	1, 3

Notion of strategy

Player I

Player II

Example: A kind of “Battle of the Sexes”



II

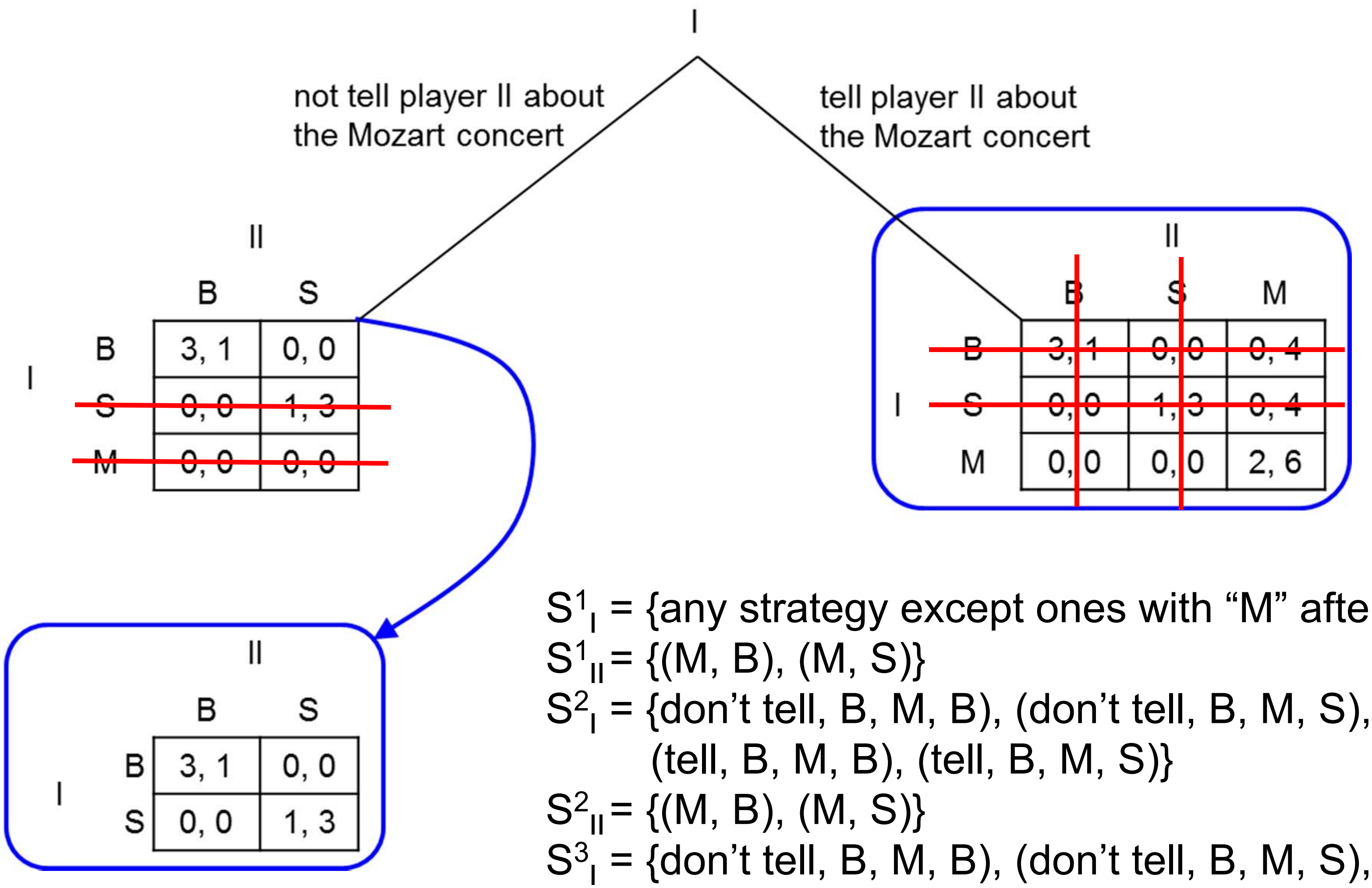
		B	S
	B	3, 1	0, 0
I	S	0, 0	1, 3
	M	0, 0	0, 0

II

		B	S	M
	B	3, 1	0, 0	0, 4
I	S	0, 0	1, 3	0, 4
	M	0, 0	0, 0	2, 6

II

		B	S
I	B	3, 1	0, 0
	S	0, 0	1, 3



S^1_I = {any strategy except ones with “M” after not tell}
S^1_{II} = {(M, B), (M, S)}
S^2_I = {don’t tell, B, M, B), (don’t tell, B, M, S),
(tell, B, M, B), (tell, B, M, S)}
S^2_{II} = {(M, B), (M, S)}
S^3_I = {don’t tell, B, M, B), (don’t tell, B, M, S),
(tell, B, M, B), (tell, B, M, S)}
S^3_{II} = {(M, B), (M, S)}

Different extensive-form rationalizable outcomes

Extensive Form Games with Unawareness

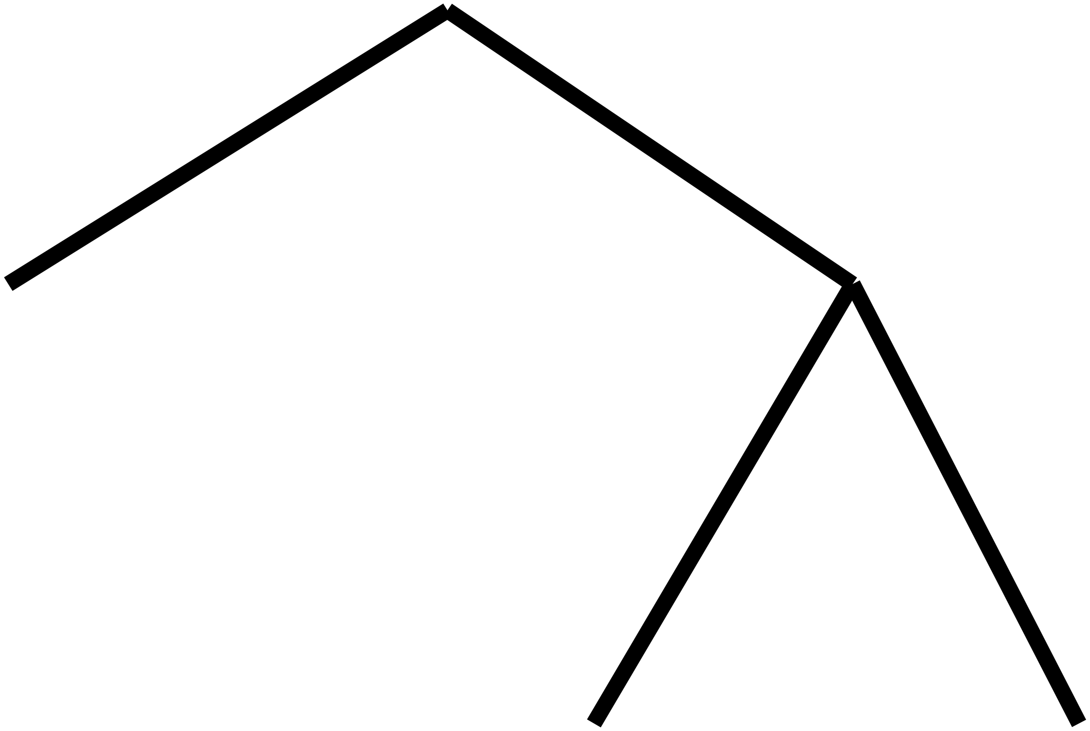
Initial building block: a standard extensive-form game with perfect information:

N_0 - decision nodes
I_n - active player(s) at node $n \in N_0$
A_n^i - actions available to player $i \in I$ at node $n \in N_0$
C_0 - chance nodes
Z_0 - terminal nodes
$(p_i^z)_{i \in I} \in \mathbb{R}^I$ - payoff vector for the players at terminal node $z \in Z_0$
$\overline{N}_0 = N_0 \cup C_0 \cup Z_0$ - the set of nodes

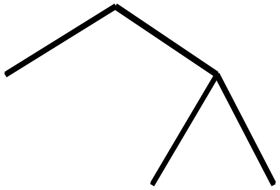
Generalized Game

Consider now a family $\mathbf{T}$ of subtrees of $\overline{N}_0$, partially ordered $(\preceq)$ by inclusion.

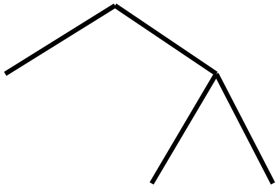
Partially Ordered Set of Trees

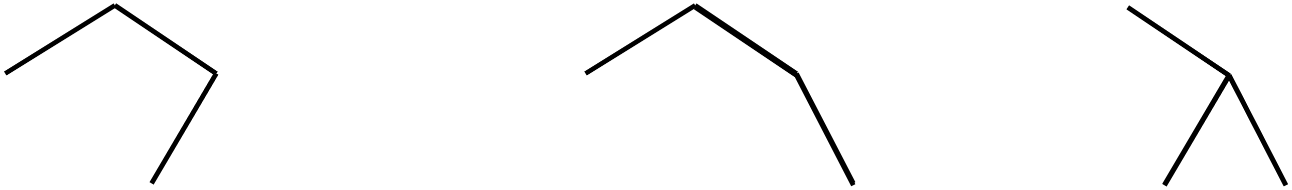


Partially Ordered Set of Trees

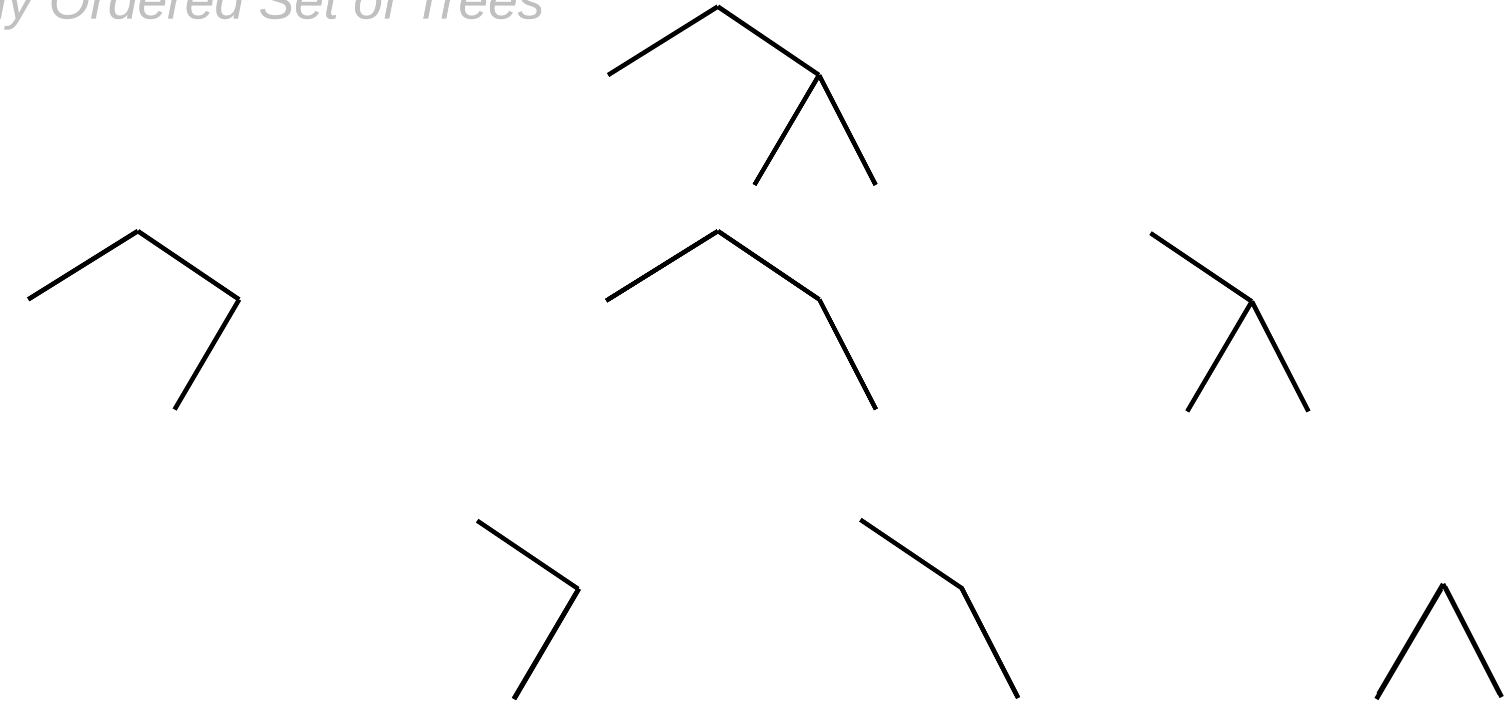


Partially Ordered Set of Trees





Partially Ordered Set of Trees



Generalized Game

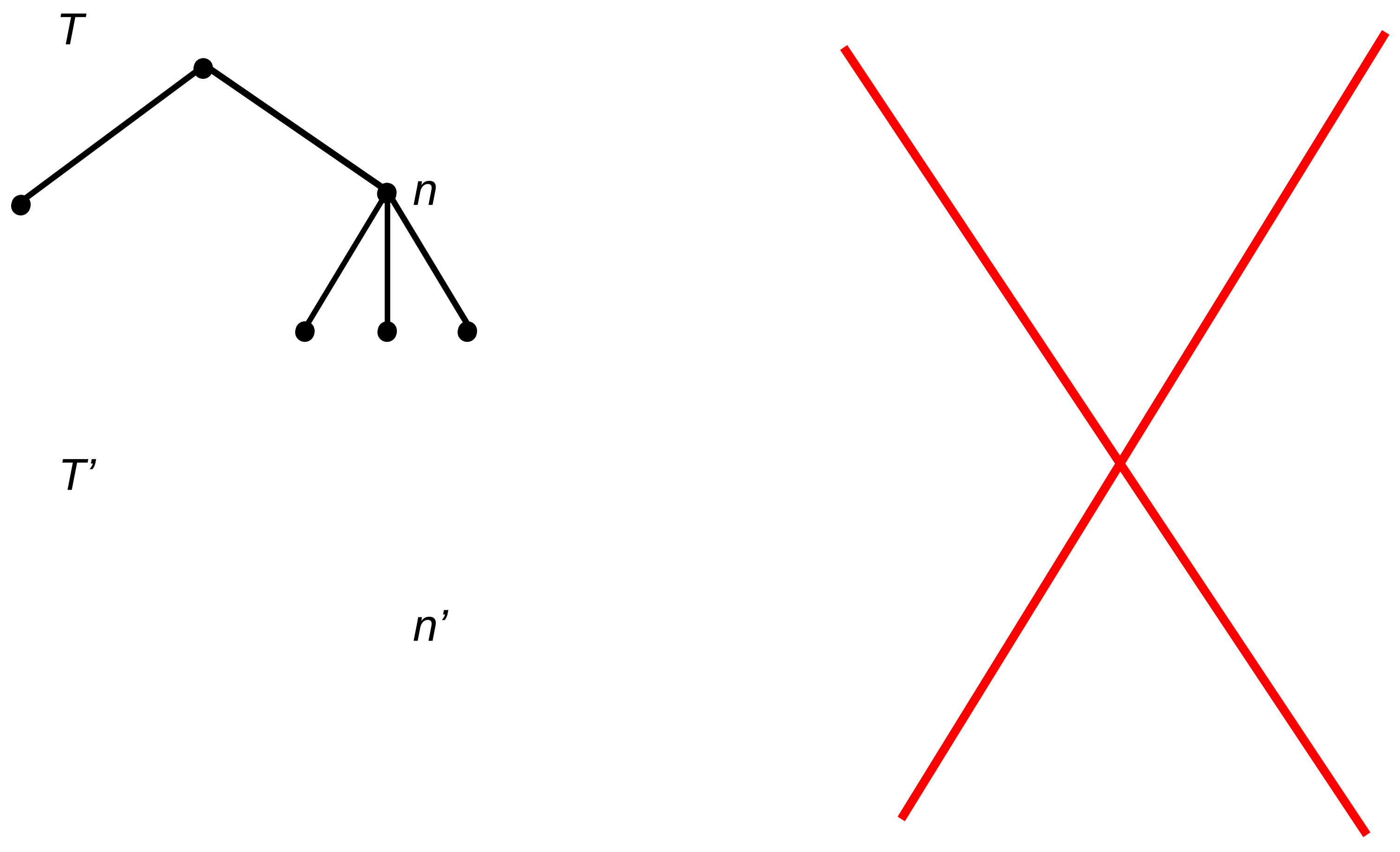
Consider now a family $\mathbf{T}$ of subtrees of $\overline{N}_0$, partially ordered $(\preceq)$ by inclusion.

In each tree $T \in \mathbf{T}$ denote by n_T the copy in T of the node $n \in N_0$ whenever the copy of n is part of the tree T.

Each subtree has the following properties:

1. All the terminal nodes in each tree $T \in \mathbf{T}$ are copies of nodes in Z_0.

1. All the terminal nodes in each tree $T \in \mathbf{T}$ are copies of nodes in Z_0.



Generalized Game

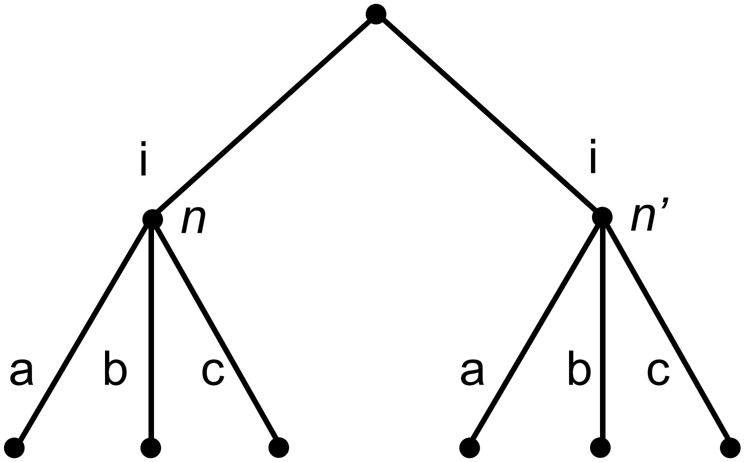
Consider now a family $\mathbf{T}$ of subtrees of $\overline{N}_0$, partially ordered ($\preceq$) by inclusion.

In each tree $T \in \mathbf{T}$ denote by n_T the copy in T of the node $n \in N_0$ whenever the copy of n is part of the tree T.

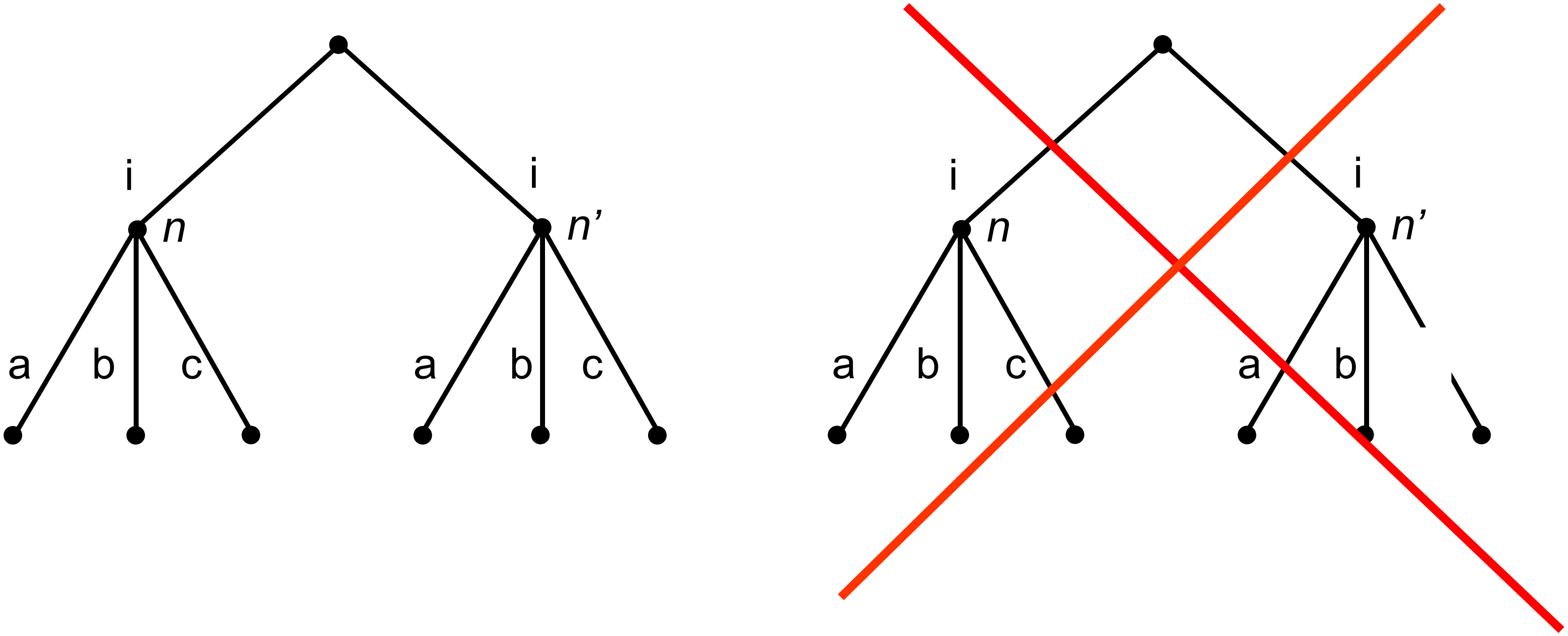
Each subtree has the following properties:

1. All the terminal nodes in each tree $T \in \mathbf{T}$ are copies of nodes in Z_0.
2. If for two decision nodes n, n' in the subtree in which some player i is active (i.e. $i \in I_n \cap I_{n'}$) we have $A^i_n \cap A^i_{n'} \neq \emptyset$, then $A^i_n = A^i_{n'}$.

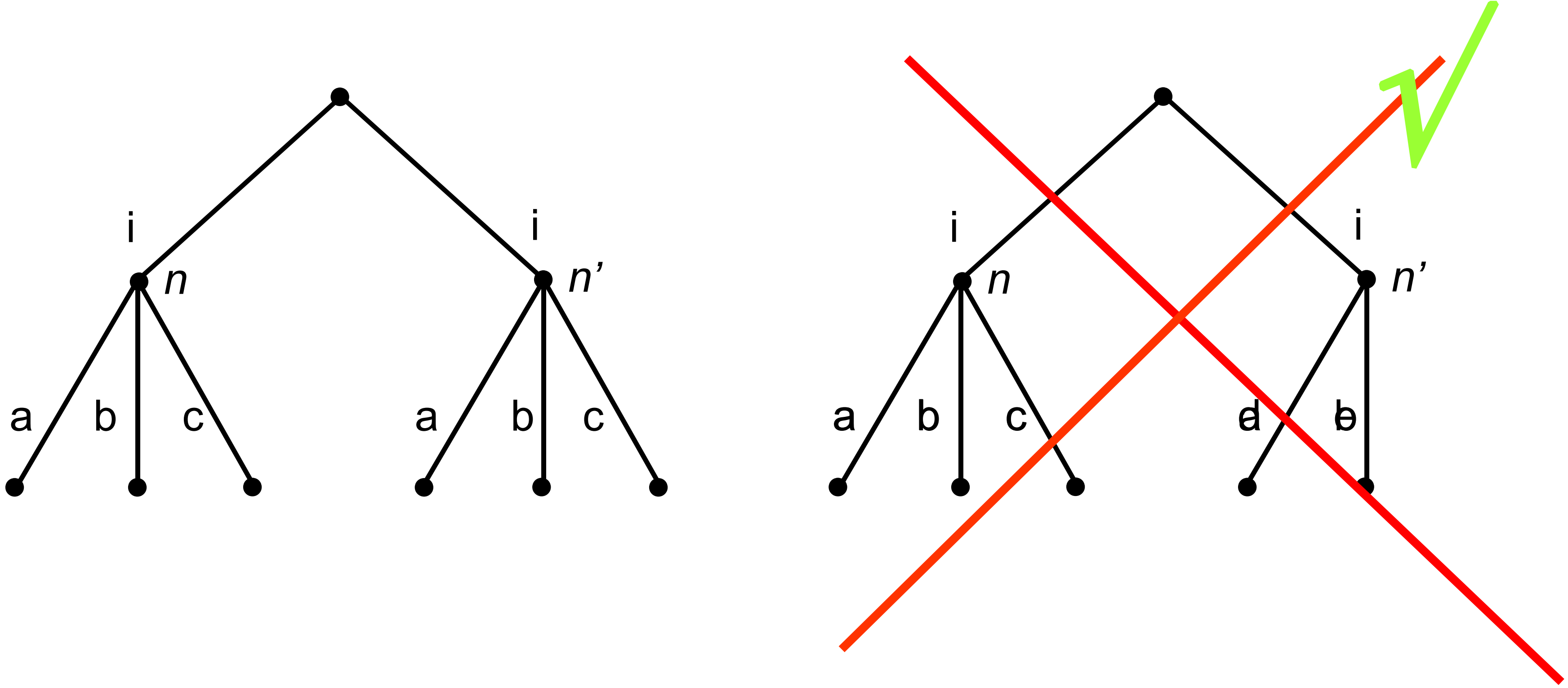
2. If for two decision nodes n, n' in the subtree in which some player i is active (i.e. $i \in I_n \cap I_{n'}$) we have $A^i_n \cap A^i_{n'} \neq \emptyset$, then $A^i_n = A^i_{n'}$.



2. If for two decision nodes n, n' in the subtree in which some player i is active (i.e. $i \in I_n \cap I_{n'}$) we have $A^i_n \cap A^i_{n'} \neq \emptyset$, then $A^i_n = A^i_{n'}$.



2. If for two decision nodes n, n' in the subtree in which some player i is active (i.e. $i \in I_n \cap I_{n'}$) we have $A^i_n \cap A^i_{n'} \neq \emptyset$, then $A^i_n = A^i_{n'}$.



N - the union of all decision nodes in all subtrees

C - the union of all chance nodes in all subtrees

Z - the union of all terminal nodes in all subtrees

$$\overline{N} = N \cup C \cup Z$$

Information sets in a generalized game

In each decision node $n \in N$, define for each active player $i \in I_n$ an information set $\pi_i(n)$ with the following properties:

I0 Confinement: $\pi_i(n) \subseteq T$ for some tree T.

I1 No delusion: If $\pi_i(n) \subseteq T_n$ then $n \in \pi_i(n)$.

I2 Introspection: If $n' \in \pi_i(n)$ then $\pi_i(n') = \pi_i(n)$.

I3 No divining of currently unimaginable paths, no expectation to forget currently conceivable paths: If $n' \in \pi_i(n) \subseteq T'$ (where $T' \in \mathbf{T}$ is a tree) and there is a path $n', \ldots, n'' \in T'$ such that $i \in I_{n'} \cap I_{n''}$ then $\pi_i(n'') \subseteq T'$.

I4 No imaginary actions: If $n' \in \pi_i(n)$ then $A_{n'}^i \subseteq A_n^i$.

I5 Distinct action names in disjoint information sets: For a subtree T, if $n, n' \in T$ and $A_n^i = A_{n'}^i$ then $\pi_i(n') = \pi_i(n)$.

I6 Perfect recall: Suppose that player i is active in two distinct nodes n_1 and n_k, and there is a path $n_1, n_2, ..., n_k$ such that at n_1 player i takes the action a_i. If $n' \in \pi_i(n_k)$, then there exists a node $n_1' \neq n'$ and a path $n_1', n_2', ..., n_\ell' = n'$ such that $\pi_i(n_1') = \pi_i(n_1)$ and at n_1' player i takes the action a_i.

Standard

I0 Confinement: $\pi_i(n) \subseteq T$ for some tree T.

I1 No delusion: If $\pi_i(n) \subseteq T_n$ then $n \in \pi_i(n)$.

I2 Introspection: If $n' \in \pi_i(n)$ then $\pi_i(n') = \pi_i(n)$.

I3 No divining of currently unimaginable paths, no expectation to forget currently conceivable paths: If $n' \in \pi_i(n) \subseteq T'$ (where $T' \in \mathbf{T}$ is a tree) and there is a path $n', \ldots, n'' \in T'$ such that $i \in I_{n'} \cap I_{n''}$ then $\pi_i(n'') \subseteq T'$.

I4 No imaginary actions: If $n' \in \pi_i(n)$ then $A^i_{n'} \subseteq A^i_n$.

I5 Distinct action names in disjoint information sets: For a subtree T, if $n, n' \in T$ and $A^i_n = A^i_{n'}$ then $\pi_i(n') = \pi_i(n)$.

I6 Perfect recall: Suppose that player i is active in two distinct nodes n_1 and n_k, and there is a path $n_1, n_2, ..., n_k$ such that at n_1 player i takes the action a_i. If $n' \in \pi_i(n_k)$, then there exists a node $n'_1 \neq n'$ and a path $n'_1, n'_2, ..., n'_\ell = n'$ such that $\pi_i(n'_1) = \pi_i(n_1)$ and at n'_1 player i takes the action a_i.

Standard
Generalized

I0 Confinement: $\pi_i(n) \subseteq T$ for some tree T.

I1 No delusion: If $\pi_i(n) \subseteq T_n$ then $n \in \pi_i(n)$.

I2 Introspection: If $n' \in \pi_i(n)$ then $\pi_i(n') = \pi_i(n)$.

I3 No divining of currently unimaginable paths, no expectation to forget currently conceivable paths: If $n' \in \pi_i(n) \subseteq T'$ (where $T' \in \mathbf{T}$ is a tree) and there is a path $n', \ldots, n'' \in T'$ such that $i \in I_{n'} \cap I_{n''}$ then $\pi_i(n'') \subseteq T'$.

I4 No imaginary actions: If $n' \in \pi_i(n)$ then $A^i_{n'} \subseteq A^i_n$.

I5 Distinct action names in disjoint information sets: For a subtree T, if $n, n' \in T$ and $A^i_n = A^i_{n'}$ then $\pi_i(n') = \pi_i(n)$.

I6 Perfect recall: Suppose that player i is active in two distinct nodes n_1 and n_k, and there is a path $n_1, n_2, ..., n_k$ such that at n_1 player i takes the action a_i. If $n' \in \pi_i(n_k)$, then there exists a node $n'_1 \neq n'$ and a path $n'_1, n'_2, ..., n'_\ell = n'$ such that $\pi_i(n'_1) = \pi_i(n_1)$ and at n'_1 player i takes the action a_i.

Standard
Generalized
New

I0 Confinement: $\pi_i(n) \subseteq T$ for some tree T.

I1 No delusion: If $\pi_i(n) \subseteq T_n$ then $n \in \pi_i(n)$.

I2 Introspection: If $n' \in \pi_i(n)$ then $\pi_i(n') = \pi_i(n)$.

I3 No divining of currently unimaginable paths, no expectation to forget currently conceivable paths: If $n' \in \pi_i(n) \subseteq T'$ (where $T' \in \mathbf{T}$ is a tree) and there is a path $n', \ldots, n'' \in T'$ such that $i \in I_{n'} \cap I_{n''}$ then $\pi_i(n'') \subseteq T'$.

I4 No imaginary actions: If $n' \in \pi_i(n)$ then $A^i_{n'} \subseteq A^i_n$.

I5 Distinct action names in disjoint information sets: For a subtree T, if $n, n' \in T$ and $A^i_n = A^i_{n'}$ then $\pi_i(n') = \pi_i(n)$.

I6 Perfect recall: Suppose that player i is active in two distinct nodes n_1 and n_k, and there is a path $n_1, n_2, ..., n_k$ such that at n_1 player i takes the action a_i. If $n' \in \pi_i(n_k)$, then there exists a node $n'_1 \neq n'$ and a path $n'_1, n'_2, ..., n'_\ell = n'$ such that $\pi_i(n'_1) = \pi_i(n_1)$ and at n'_1 player i takes the action a_i.

I0 Confinement: $\pi_i(n) \subseteq T$ for some tree T.

I1 No delusion: If $\pi_i(n) \subseteq T_n$ then $n \in \pi_i(n)$.

I2 Introspection: If $n' \in \pi_i(n)$ then $\pi_i(n') = \pi_i(n)$.

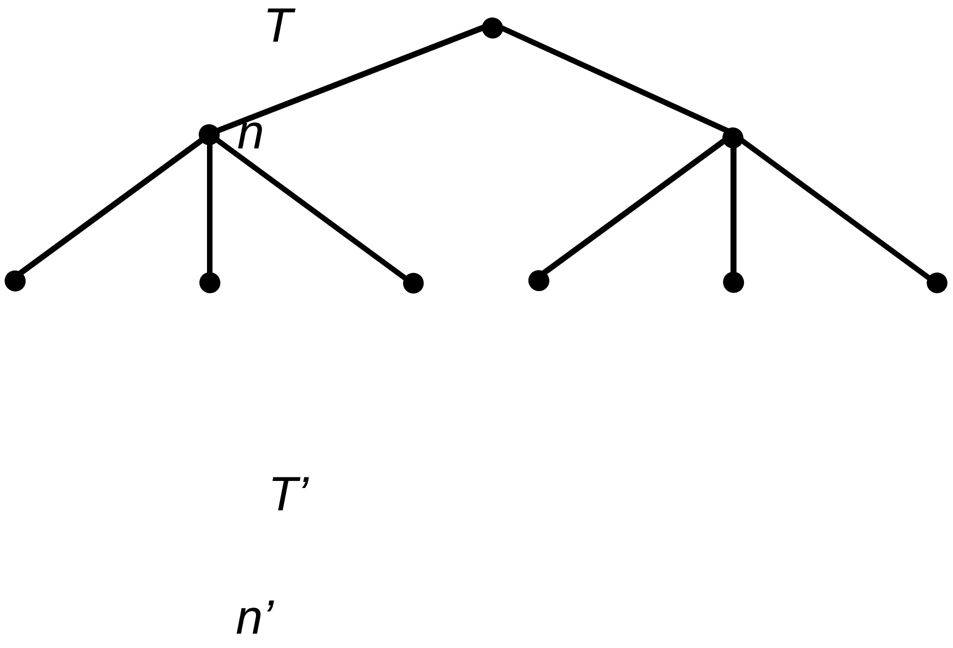
I3 No divining of currently unimaginable paths, no expectation to forget currently conceivable paths: If $n' \in \pi_i(n) \subseteq T'$ (where $T' \in \mathbf{T}$ is a tree) and there is a path $n', \ldots, n'' \in T'$ such that $i \in I_{n'} \cap I_{n''}$ then $\pi_i(n'') \subseteq T'$.

I4 No imaginary actions: If $n' \in \pi_i(n)$ then $A^i_{n'} \subseteq A^i_n$.

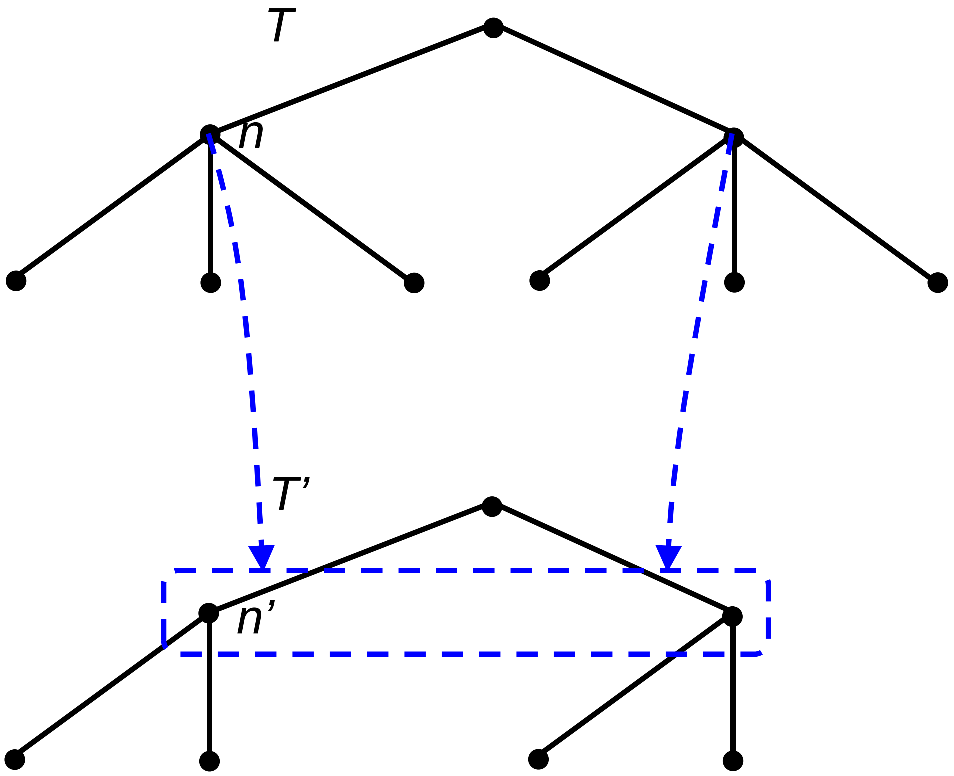
I5 Distinct action names in disjoint information sets: For a subtree T, if $n, n' \in T$ and $A^i_n = A^i_{n'}$ then $\pi_i(n') = \pi_i(n)$.

I6 Perfect recall: Suppose that player i is active in two distinct nodes n_1 and n_k, and there is a path $n_1, n_2, ..., n_k$ such that at n_1 player i takes the action a_i. If $n' \in \pi_i(n_k)$, then there exists a node $n'_1 \neq n'$ and a path $n'_1, n'_2, ..., n'_\ell = n'$ such that $\pi_i(n'_1) = \pi_i(n_1)$ and at n'_1 player i takes the action a_i.

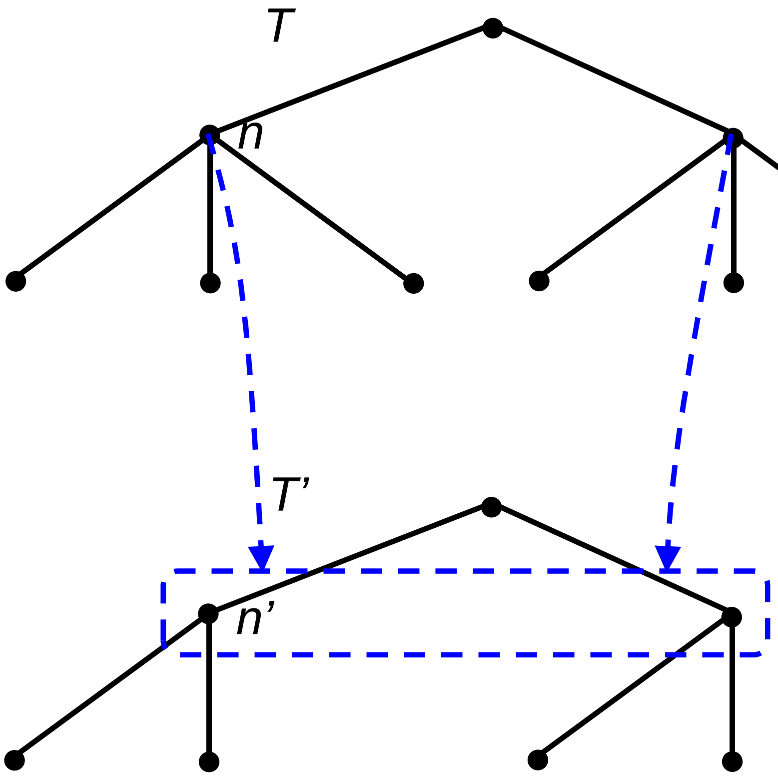
I0 Confinement: $\pi_i(n) \subseteq T$ for some subtree T.

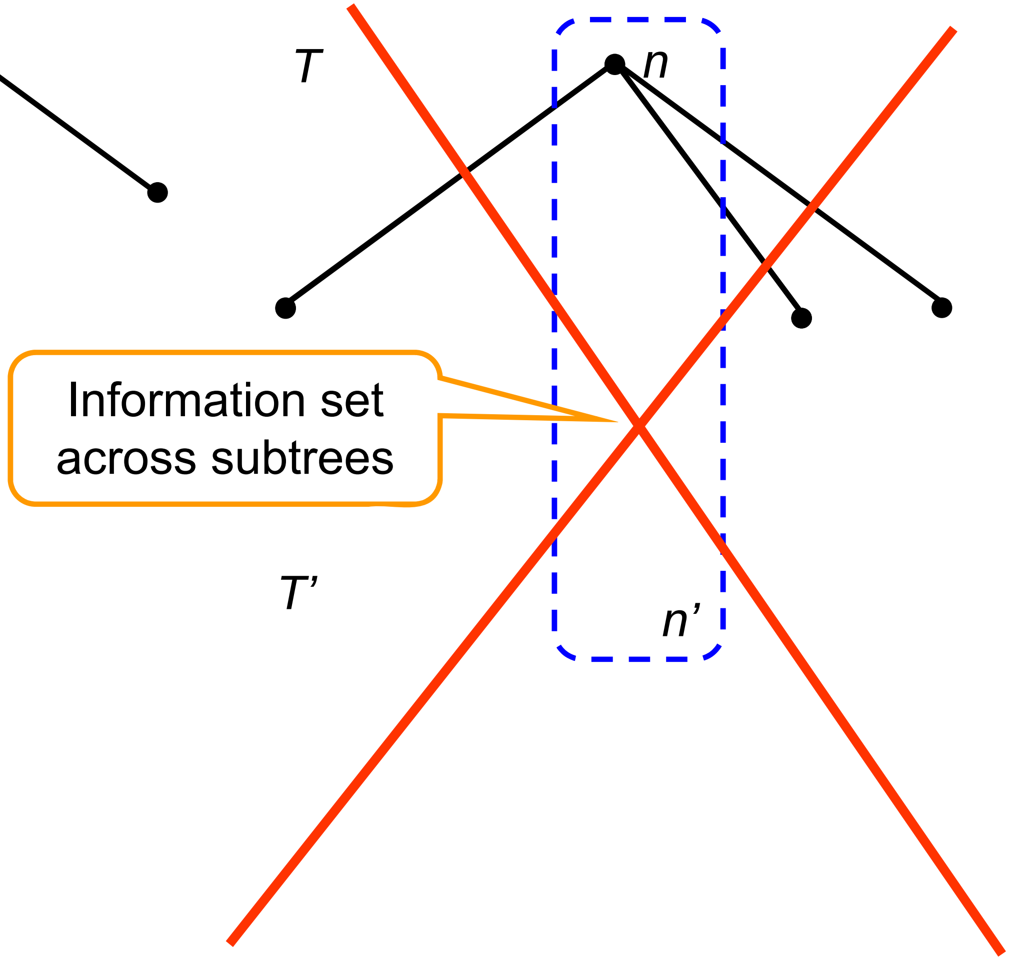


I0 Confinement: $\pi_i(n) \subseteq T$ for some subtree T.



IO Confinement: $\pi_i(n) \subseteq T$ for some subtree T.





Information set across subtrees

I0 Confinement: $\pi_i(n) \subseteq T$ for some tree T.

I1 No delusion: If $\pi_i(n) \subseteq T_n$ then $n \in \pi_i(n)$.

I2 Introspection: If $n' \in \pi_i(n)$ then $\pi_i(n') = \pi_i(n)$.

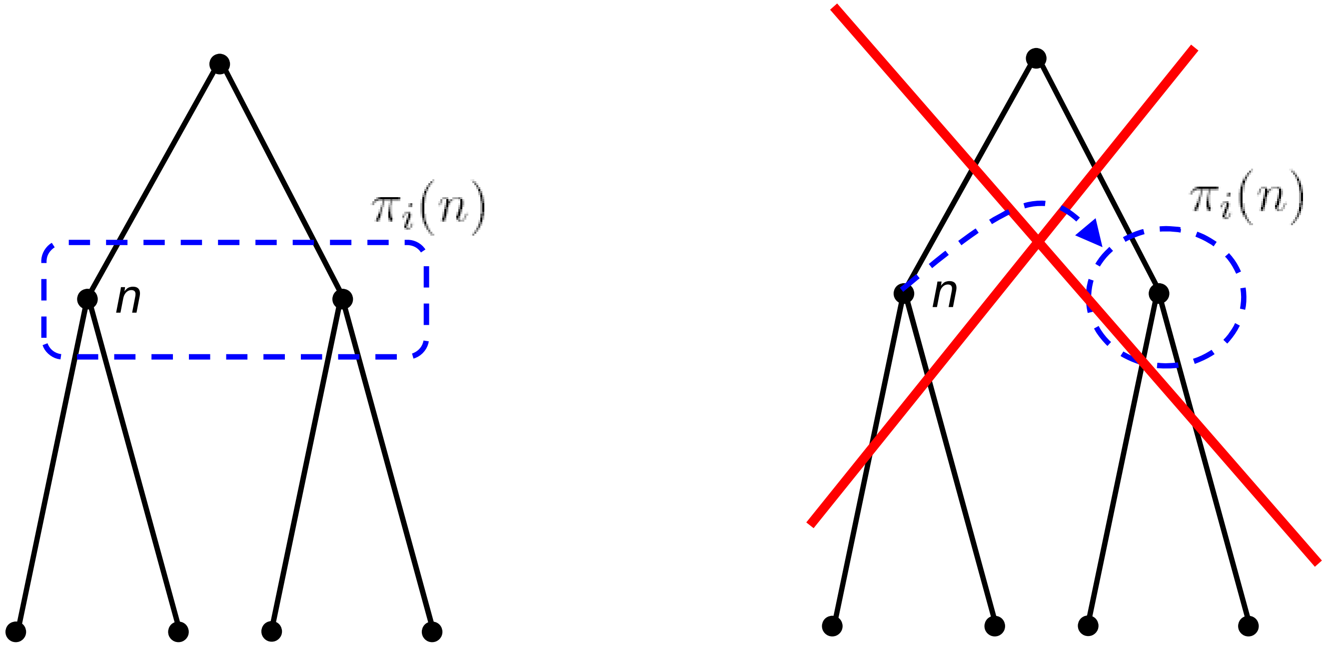
I3 No divining of currently unimaginable paths, no expectation to forget currently conceivable paths: If $n' \in \pi_i(n) \subseteq T'$ (where $T' \in \mathbf{T}$ is a tree) and there is a path $n', \ldots, n'' \in T'$ such that $i \in I_{n'} \cap I_{n''}$ then $\pi_i(n'') \subseteq T'$.

I4 No imaginary actions: If $n' \in \pi_i(n)$ then $A^i_{n'} \subseteq A^i_n$.

I5 Distinct action names in disjoint information sets: For a subtree T, if $n, n' \in T$ and $A^i_n = A^i_{n'}$ then $\pi_i(n') = \pi_i(n)$.

I6 Perfect recall: Suppose that player i is active in two distinct nodes n_1 and n_k, and there is a path $n_1, n_2, ..., n_k$ such that at n_1 player i takes the action a_i. If $n' \in \pi_i(n_k)$, then there exists a node $n'_1 \neq n'$ and a path $n'_1, n'_2, ..., n'_\ell = n'$ such that $\pi_i(n'_1) = \pi_i(n_1)$ and at n'_1 player i takes the action a_i.

I1 No delusion: If $\pi_i(n) \subseteq T_n$ then $n \in \pi_i(n)$.



I0 Confinement: $\pi_i(n) \subseteq T$ for some tree T.

I1 No delusion: If $\pi_i(n) \subseteq T_n$ then $n \in \pi_i(n)$.

I2 Introspection: If $n' \in \pi_i(n)$ then $\pi_i(n') = \pi_i(n)$.

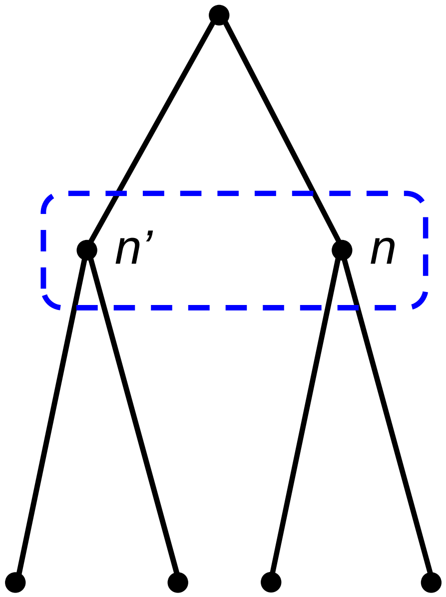
I3 No divining of currently unimaginable paths, no expectation to forget currently conceivable paths: If $n' \in \pi_i(n) \subseteq T'$ (where $T' \in \mathbf{T}$ is a tree) and there is a path $n', \ldots, n'' \in T'$ such that $i \in I_{n'} \cap I_{n''}$ then $\pi_i(n'') \subseteq T'$.

I4 No imaginary actions: If $n' \in \pi_i(n)$ then $A^i_{n'} \subseteq A^i_n$.

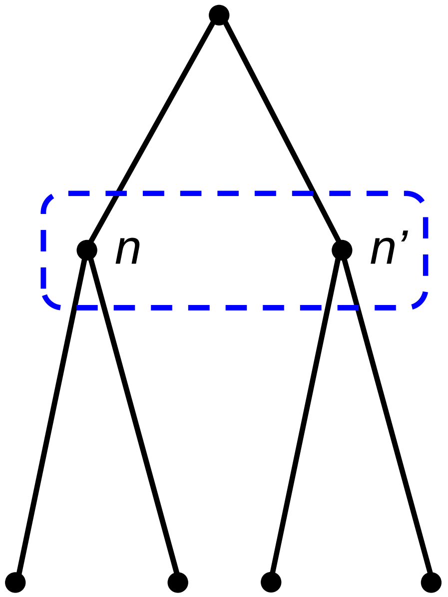
I5 Distinct action names in disjoint information sets: For a subtree T, if $n, n' \in T$ and $A^i_n = A^i_{n'}$ then $\pi_i(n') = \pi_i(n)$.

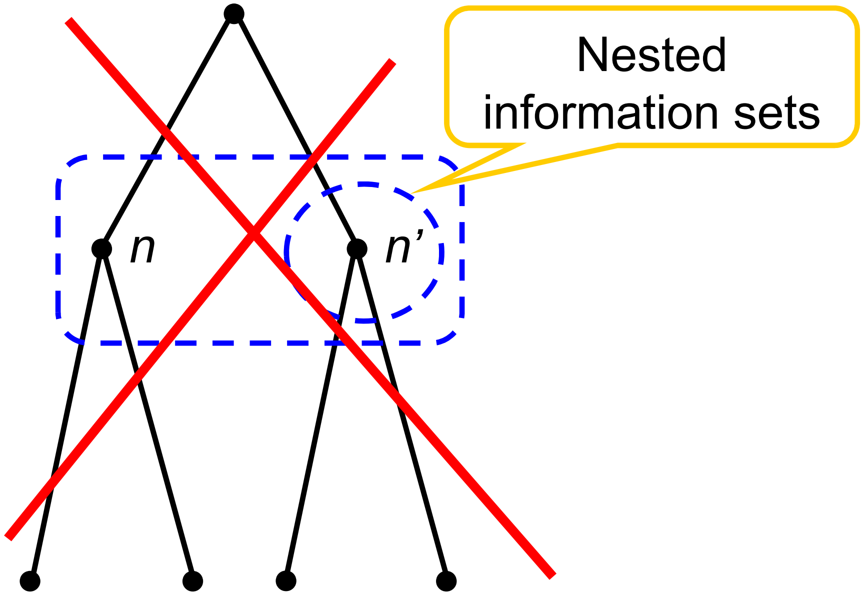
I6 Perfect recall: Suppose that player i is active in two distinct nodes n_1 and n_k, and there is a path $n_1, n_2, ..., n_k$ such that at n_1 player i takes the action a_i. If $n' \in \pi_i(n_k)$, then there exists a node $n'_1 \neq n'$ and a path $n'_1, n'_2, ..., n'_\ell = n'$ such that $\pi_i(n'_1) = \pi_i(n_1)$ and at n'_1 player i takes the action a_i.

I2 Introspection: If $n' \in \pi_i(n)$ then $\pi_i(n') = \pi_i(n)$.



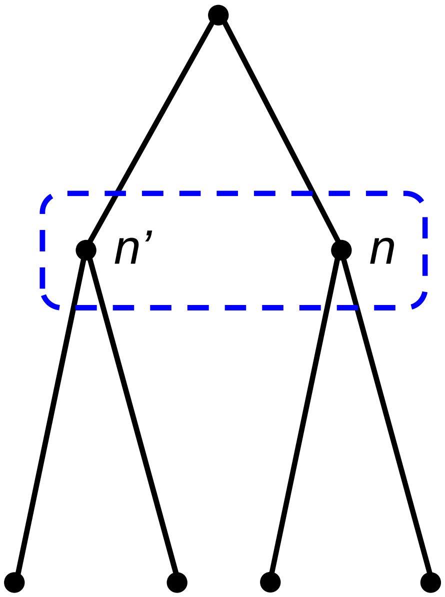
I2 Introspection: If $n' \in \pi_i(n)$ then $\pi_i(n') = \pi_i(n)$.

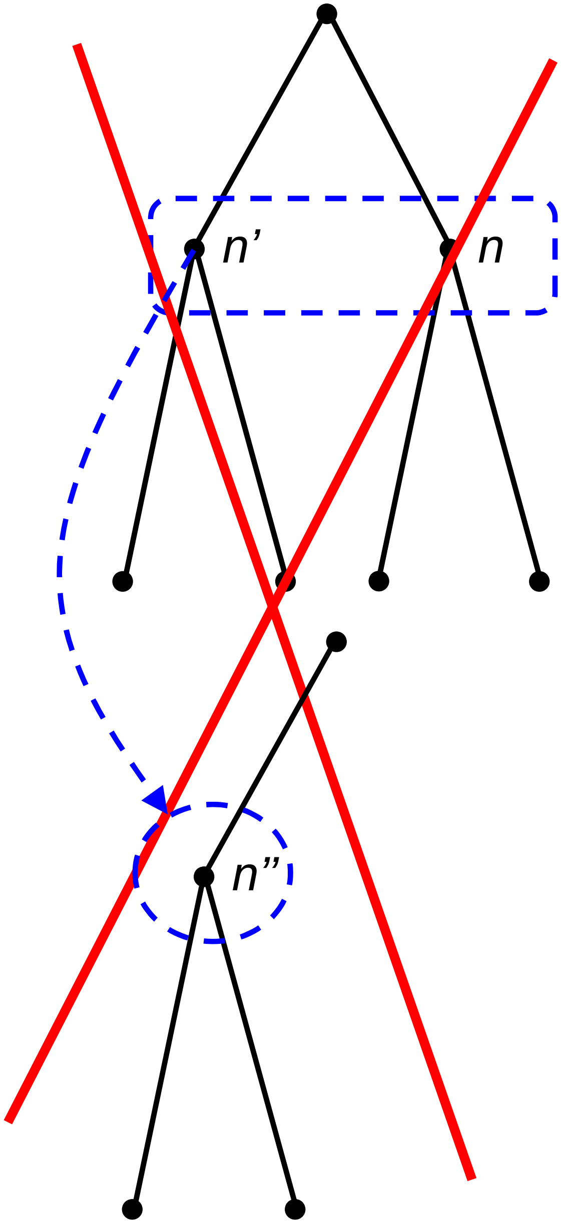




Nested information sets

I2 Introspection: If $n' \in \pi_i(n)$ then $\pi_i(n') = \pi_i(n)$.





I0 Confinement: $\pi_i(n) \subseteq T$ for some tree T.

I1 No delusion: If $\pi_i(n) \subseteq T_n$ then $n \in \pi_i(n)$.

I2 Introspection: If $n' \in \pi_i(n)$ then $\pi_i(n') = \pi_i(n)$.

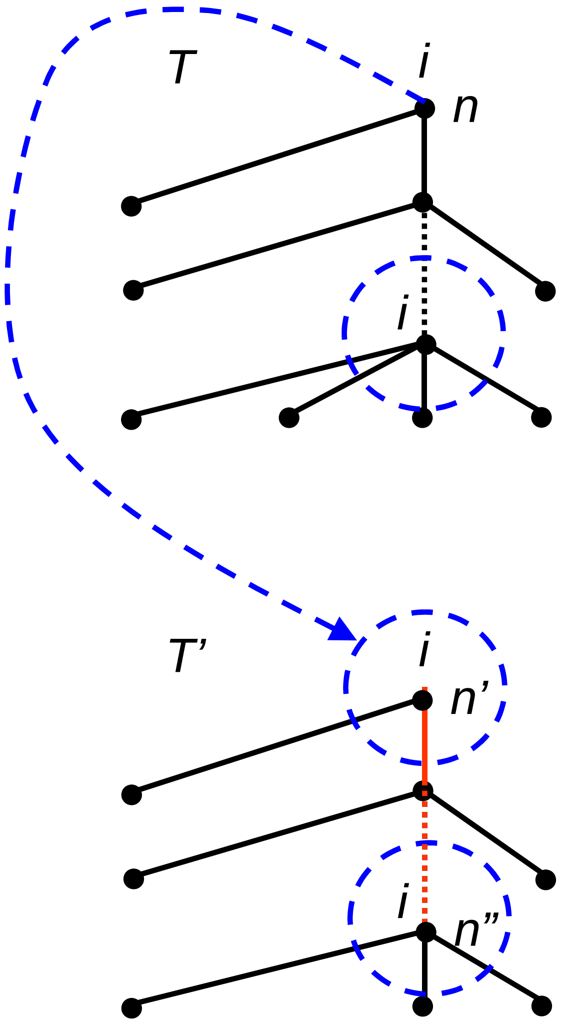
I3 No divining of currently unimaginable paths, no expectation to forget currently conceivable paths: If $n' \in \pi_i(n) \subseteq T'$ (where $T' \in \mathbf{T}$ is a tree) and there is a path $n', \ldots, n'' \in T'$ such that $i \in I_{n'} \cap I_{n''}$ then $\pi_i(n'') \subseteq T'$.

I4 No imaginary actions: If $n' \in \pi_i(n)$ then $A^i_{n'} \subseteq A^i_n$.

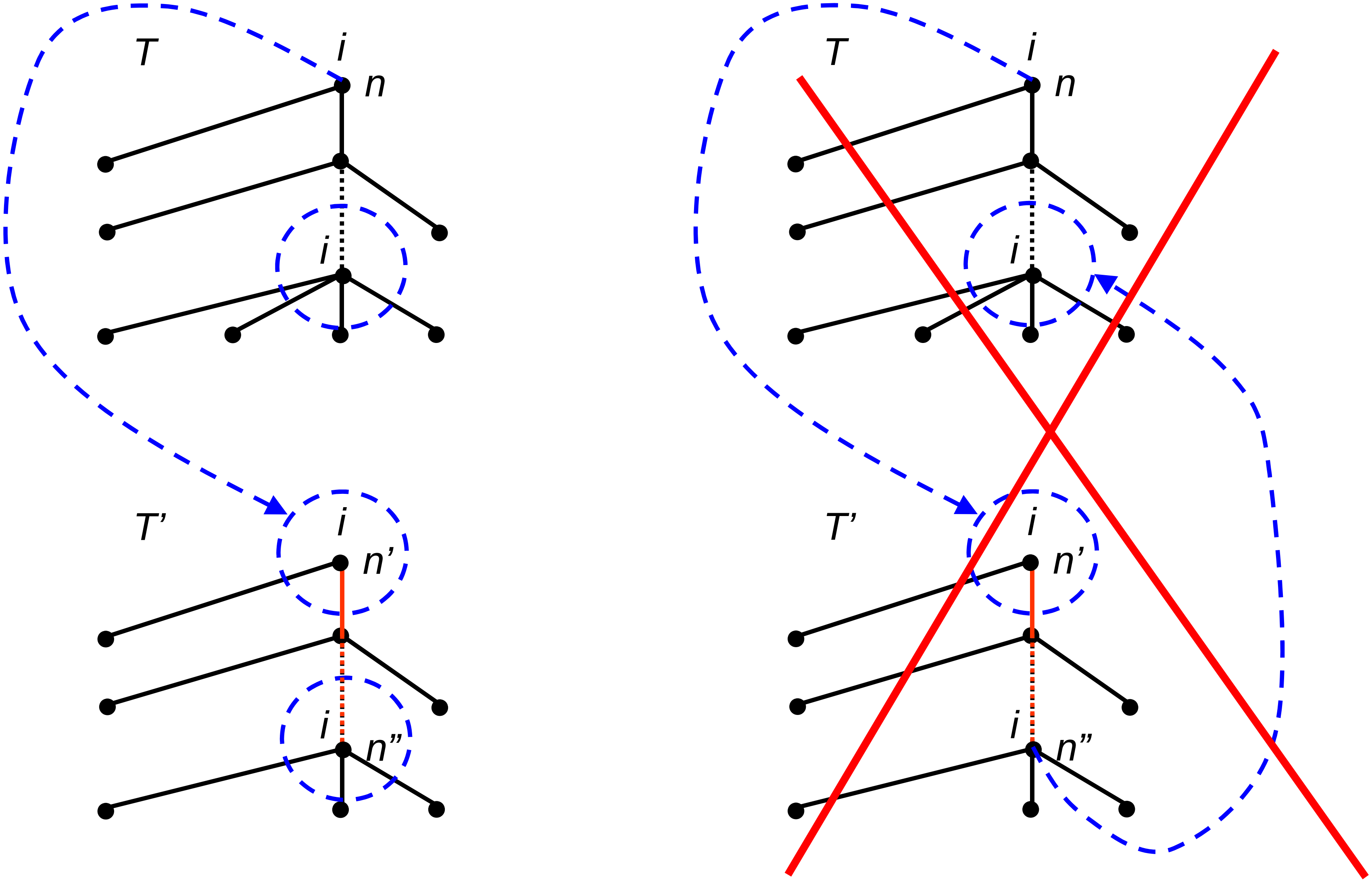
I5 Distinct action names in disjoint information sets: For a subtree T, if $n, n' \in T$ and $A^i_n = A^i_{n'}$ then $\pi_i(n') = \pi_i(n)$.

I6 Perfect recall: Suppose that player i is active in two distinct nodes n_1 and n_k, and there is a path $n_1, n_2, ..., n_k$ such that at n_1 player i takes the action a_i. If $n' \in \pi_i(n_k)$, then there exists a node $n'_1 \neq n'$ and a path $n'_1, n'_2, ..., n'_\ell = n'$ such that $\pi_i(n'_1) = \pi_i(n_1)$ and at n'_1 player i takes the action a_i.

I3 No divining of currently unimaginable paths, no expectation to forget currently conceivable paths: If $n' \in \pi_i(n) \subseteq T'$ (where $T' \in \mathbf{T}$ is a tree) and there is a path $n', \ldots, n'' \in T'$ such that $i \in I_{n'} \cap I_{n''}$ then $\pi_i(n'') \subseteq T'$.



I3 No divining of currently unimaginable paths, no expectation to forget currently conceivable paths: If $n' \in \pi_i(n) \subseteq T'$ (where $T' \in \mathbf{T}$ is a tree) and there is a path $n', \ldots, n'' \in T'$ such that $i \in I_{n'} \cap I_{n''}$ then $\pi_i(n'') \subseteq T'$.



I0 Confinement: $\pi_i(n) \subseteq T$ for some tree T.

I1 No delusion: If $\pi_i(n) \subseteq T_n$ then $n \in \pi_i(n)$.

I2 Introspection: If $n' \in \pi_i(n)$ then $\pi_i(n') = \pi_i(n)$.

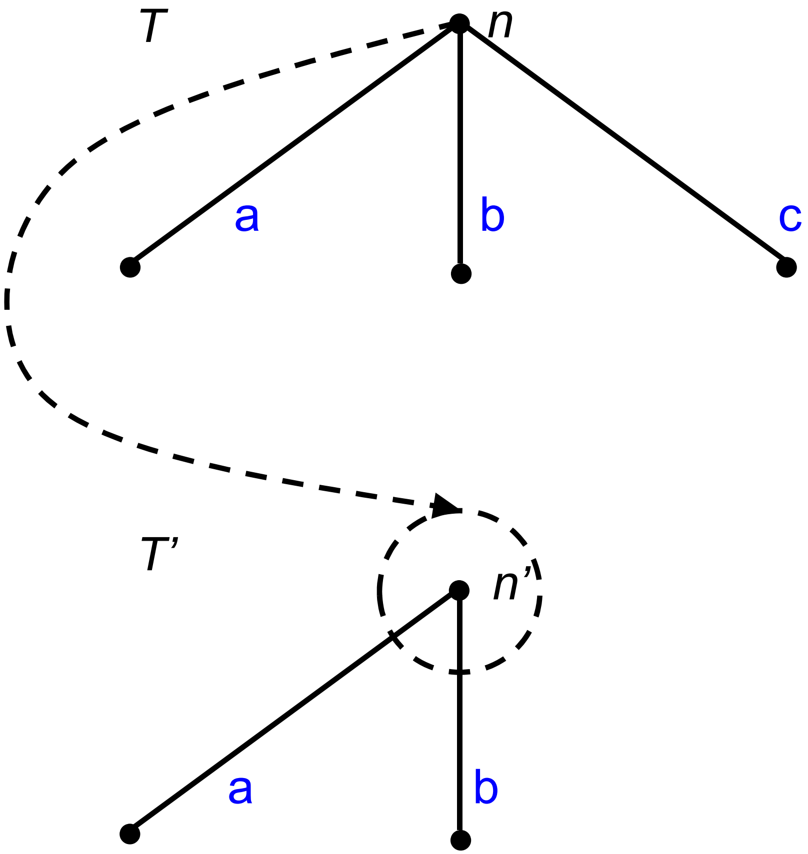
I3 No divining of currently unimaginable paths, no expectation to forget currently conceivable paths: If $n' \in \pi_i(n) \subseteq T'$ (where $T' \in \mathbf{T}$ is a tree) and there is a path $n', \ldots, n'' \in T'$ such that $i \in I_{n'} \cap I_{n''}$ then $\pi_i(n'') \subseteq T'$.

I4 No imaginary actions: If $n' \in \pi_i(n)$ then $A^i_{n'} \subseteq A^i_n$.

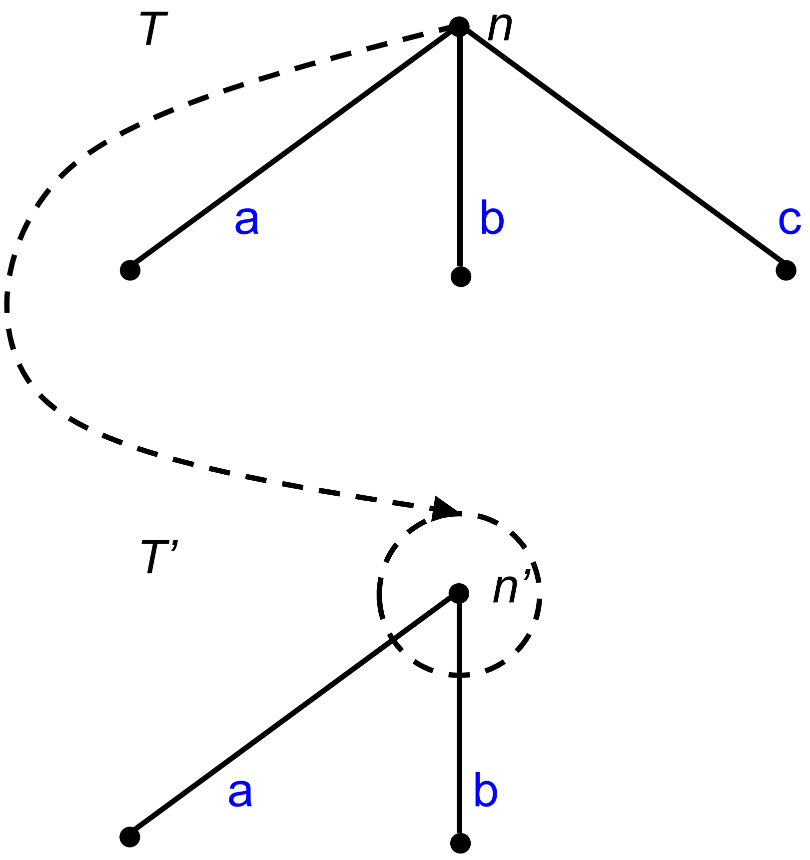
I5 Distinct action names in disjoint information sets: For a subtree T, if $n, n' \in T$ and $A^i_n = A^i_{n'}$ then $\pi_i(n') = \pi_i(n)$.

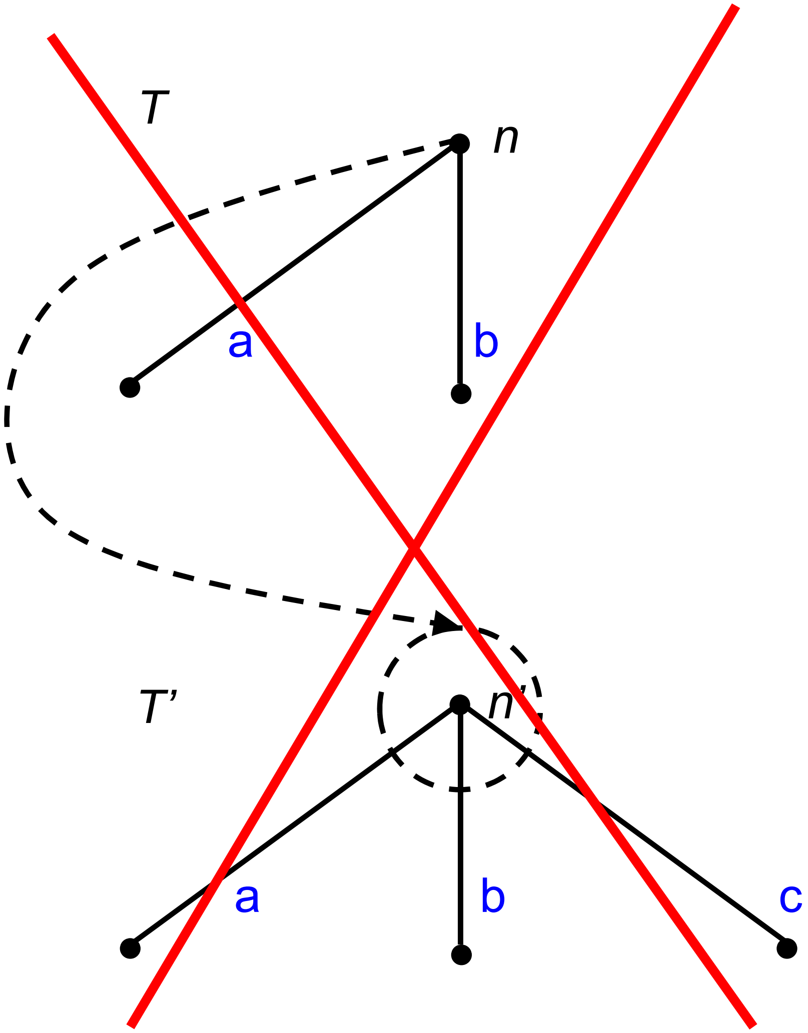
I6 Perfect recall: Suppose that player i is active in two distinct nodes n_1 and n_k, and there is a path $n_1, n_2, ..., n_k$ such that at n_1 player i takes the action a_i. If $n' \in \pi_i(n_k)$, then there exists a node $n'_1 \neq n'$ and a path $n'_1, n'_2, ..., n'_\ell = n'$ such that $\pi_i(n'_1) = \pi_i(n_1)$ and at n'_1 player i takes the action a_i.

I4 No imaginary actions: If $n' \in \pi_i(n)$ then $A^i_{n'} \subseteq A^i_n$.



I4 No imaginary actions: If $n' \in \pi_i(n)$ then $A^i_{n'} \subseteq A^i_n$.





I0 Confinement: $\pi_i(n) \subseteq T$ for some tree T.

I1 No delusion: If $\pi_i(n) \subseteq T_n$ then $n \in \pi_i(n)$.

I2 Introspection: If $n' \in \pi_i(n)$ then $\pi_i(n') = \pi_i(n)$.

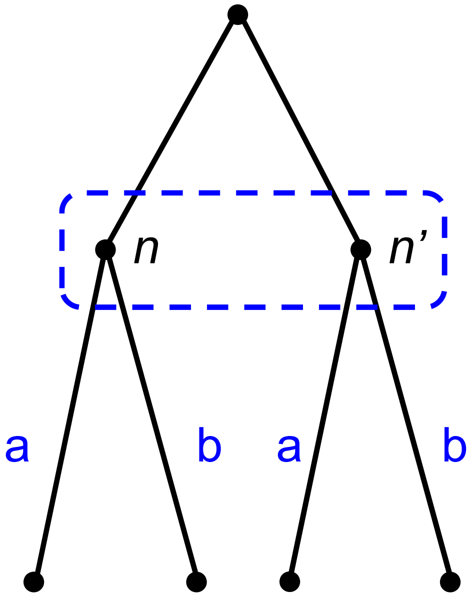
I3 No divining of currently unimaginable paths, no expectation to forget currently conceivable paths: If $n' \in \pi_i(n) \subseteq T'$ (where $T' \in \mathbf{T}$ is a tree) and there is a path $n', \ldots, n'' \in T'$ such that $i \in I_{n'} \cap I_{n''}$ then $\pi_i(n'') \subseteq T'$.

I4 No imaginary actions: If $n' \in \pi_i(n)$ then $A^i_{n'} \subseteq A^i_n$.

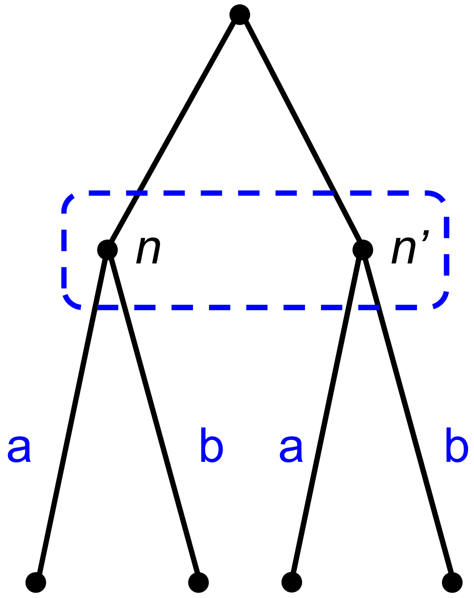
I5 Distinct action names in disjoint information sets: For a subtree T, if $n, n' \in T$ and $A^i_n = A^i_{n'}$ then $\pi_i(n') = \pi_i(n)$.

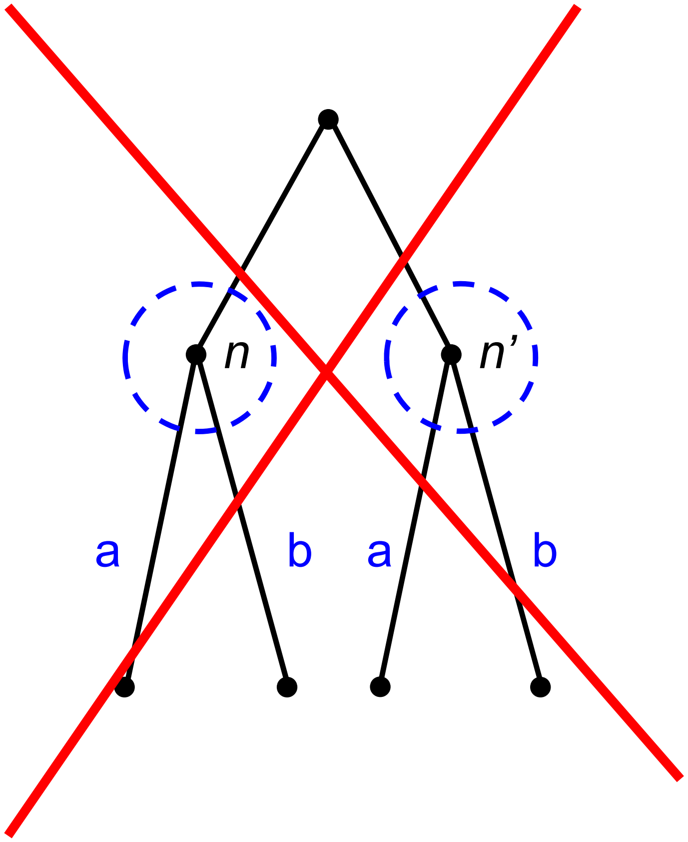
I6 Perfect recall: Suppose that player i is active in two distinct nodes n_1 and n_k, and there is a path $n_1, n_2, ..., n_k$ such that at n_1 player i takes the action a_i. If $n' \in \pi_i(n_k)$, then there exists a node $n'_1 \neq n'$ and a path $n'_1, n'_2, ..., n'_\ell = n'$ such that $\pi_i(n'_1) = \pi_i(n_1)$ and at n'_1 player i takes the action a_i.

I5 Distinct action names in disjoint information sets: For a subtree T, if $n, n' \in T$ and $A_n^i = A_{n'}^i$, then $\pi_i(n') = \pi_i(n)$.



I5 Distinct action names in disjoint information sets: For a subtree T, if $n, n' \in T$ and $A^i_n = A^i_{n'}$, then $\pi_i(n') = \pi_i(n)$.





I0 Confinement: $\pi_i(n) \subseteq T$ for some tree T.

I1 No delusion: If $\pi_i(n) \subseteq T_n$ then $n \in \pi_i(n)$.

I2 Introspection: If $n' \in \pi_i(n)$ then $\pi_i(n') = \pi_i(n)$.

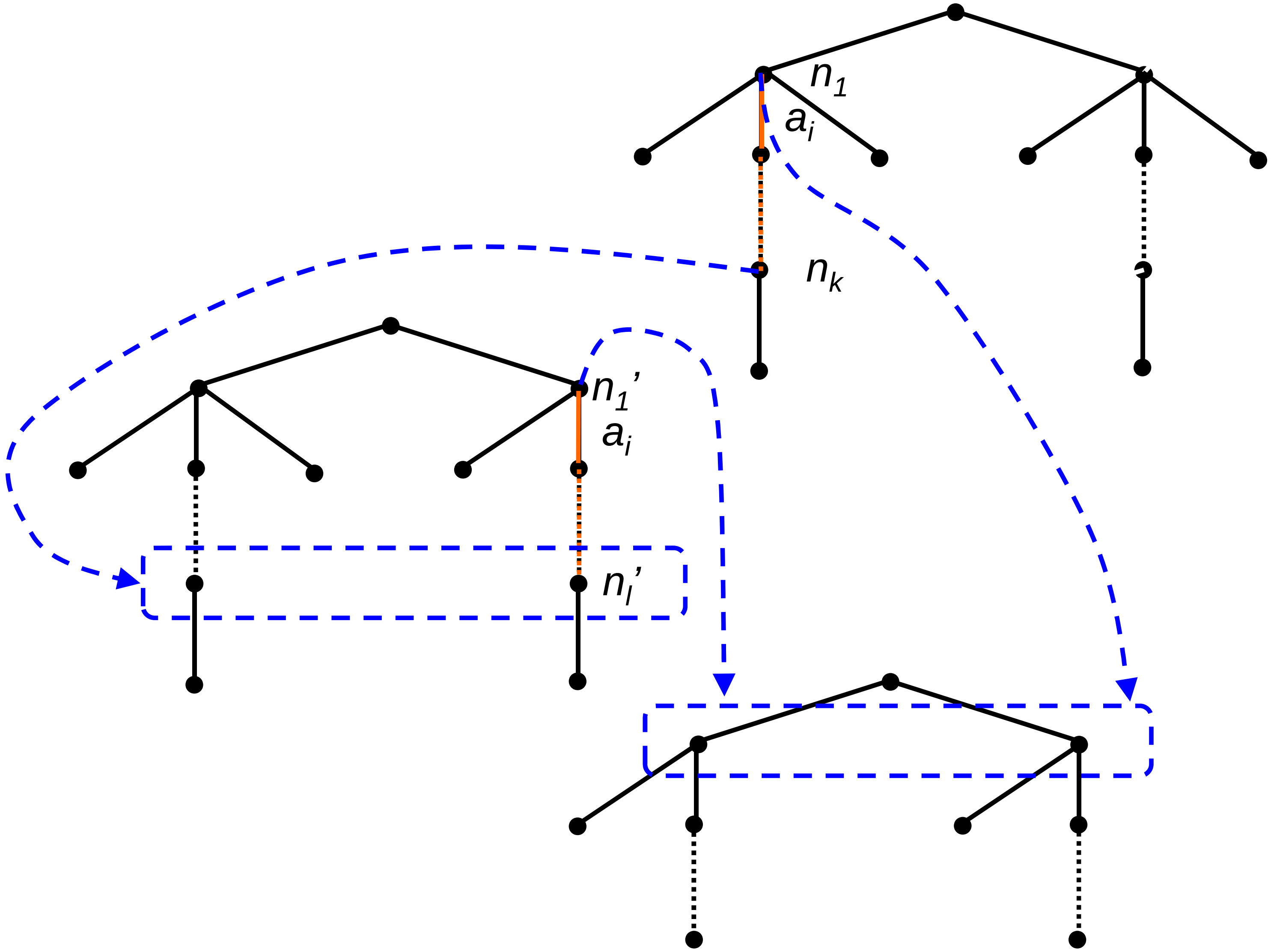
I3 No divining of currently unimaginable paths, no expectation to forget currently conceivable paths: If $n' \in \pi_i(n) \subseteq T'$ (where $T' \in \mathbf{T}$ is a tree) and there is a path $n', \ldots, n'' \in T'$ such that $i \in I_{n'} \cap I_{n''}$ then $\pi_i(n'') \subseteq T'$.

I4 No imaginary actions: If $n' \in \pi_i(n)$ then $A^i_{n'} \subseteq A^i_n$.

I5 Distinct action names in disjoint information sets: For a subtree T, if $n, n' \in T$ and $A^i_n = A^i_{n'}$ then $\pi_i(n') = \pi_i(n)$.

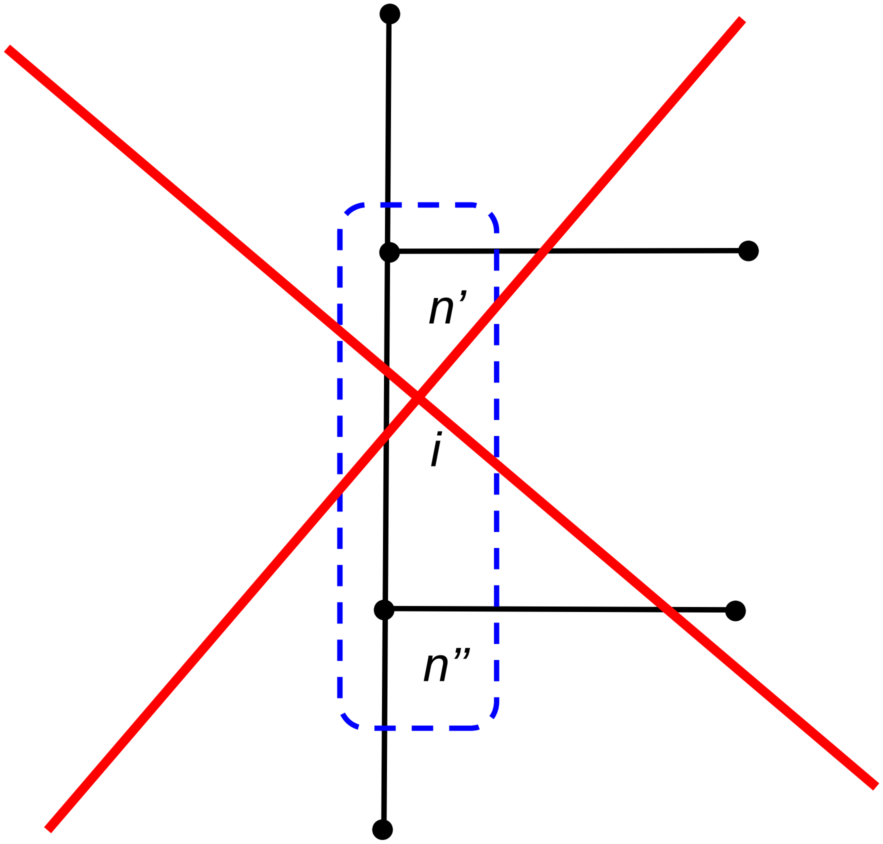
I6 Perfect recall: Suppose that player i is active in two distinct nodes n_1 and n_k, and there is a path $n_1, n_2, ..., n_k$ such that at n_1 player i takes the action a_i. If $n' \in \pi_i(n_k)$, then there exists a node $n'_1 \neq n'$ and a path $n'_1, n'_2, ..., n'_\ell = n'$ such that $\pi_i(n'_1) = \pi_i(n_1)$ and at n'_1 player i takes the action a_i.

I6 Perfect recall: Suppose that player i is active in two distinct nodes n_1 and n_k, and there is a path $n_1, n_2, ..., n_k$ such that at n_1 player i takes the action a_i. If $n' \in \pi_i(n_k)$, then there exists a node $n'_1 \neq n'$ and a path $n'_1, n'_2, ..., n'_\ell = n'$ such that $\pi_i(n'_1) = \pi_i(n_1)$ and at n'_1 player i takes the action a_i.



Implied Property: If $n', n'' \in h_i$ where $h_i = \pi_i(n)$ is an information set, then $A^i_{n'} = A^i_{n''}$.

Another Implied Property: Properties I0, I1, I2 and I6 imply no absent-mindedness: No information set h_i contains two distinct nodes n, n' on some path in some tree.



For two information sets h_i, h'_i in a given tree T, we say that h_i precedes h'_i (denoted by $h_i \rightsquigarrow h'_i$) if for every $n' \in h'_i$ there is a path $n, ..., n'$ such that $n \in h_i$.

Denote by H_i the set of i's information sets in all subtrees. For an information set $h_i \in H_i$, we denote by T_{h_i} the tree containing h_i.

The perfect recall properties (No Absentmindedness & No Forgetfulness) guarantee that with the precedence relation $\rightsquigarrow$ player i's information sets H_i form an *arborescence*: For every information set $h'_i \in H_i$, the information sets preceding it $\{h_i \in H_i : h_i \rightsquigarrow h'_i\}$ are totally ordered by $\rightsquigarrow$.

For trees $T, T' \in \mathbf{T}$ we denote $T \rightarrowtail T'$ whenever for some node $n \in T$ and some player $i \in I_n$ it is the case that $\pi_i(n) \subseteq T'$. Denote by $\hookrightarrow$ the transitive closure of $\rightarrowtail$. That is, $T \hookrightarrow T''$ iff there is a sequence of subtrees $T, T', \ldots, T''$ satisfying $T \rightarrowtail T' \rightarrowtail \cdots \rightarrowtail T''$.

Generalized Extensive-Form Games

A generalized extensive-form game G consists of a partially ordered set $\mathbf{T}$ of subtrees of a tree $\bar{N}_0$ satisfying properties 1-2above, along with information sets $\pi_i(n)$ for every $n \in T, T \in \mathbf{T}$ and $i \in I_n,$ satisfying properties I0-I6 above.

For every tree $T \in \mathbf{T}$, the *T-partial game* is the partially ordered set of trees including T and all trees T' in G satisfying $T \hookrightarrow T'$, with information sets as defined in G. A T-partial game is a generalized game, i.e. it satisfies all properties 1-2 and I0-I6.

We denote by H_i^T the set of i's information sets in the T-partial game T.

Extensive-form Rationalizability

Pearce (1984) defined an algorithm for iterative elimination of strategies in extensive-form games, which captures a notion of forward induction.

In generic perfect-information games, rationalizable strategy profiles yield the backward induction outcome but not necessarily subgame-perfect equilibrium strategies (Reny 1992, Battigalli, 1997).

We generalize extensive-form rationalizability to extensive games with unawareness.

Strategies

A strategy s_i of player i assigns to each information set H_i an action.

In general, a strategy cannot be conceived as an ex ante plan of action.

Rather, it is a list of answers to the hypothetical questions "what would the player do if h_i were the set of nodes she considered as possible?" for $h_i \in H_i$.

Such a question about the information set $h'_i \in H_i^{T'}$ may not even be meaningful to the player if it were asked at a different information set $h_i \in H_i^T$ when $T \not\leftrightarrow T'$.

Belief Systems

A *belief system* of player i

$$b_i = (b_i(h_i))_{h_i \in H_i} \in \prod_{h_i \in H_i} \Delta\left(S_{-i}^{T_{h_i}}\right)$$

is a profile of beliefs - a belief $b_i(h_i) \in \Delta\left(S_{-i}^{T_{h_i}}\right)$ about the other players' strategies in the T_{h_i}-partial game, for each information set $h_i \in H_i$, with the following properties

- $b_i(h_i)$ reaches h_i, i.e. $b_i(h_i)$ assigns probability 1 to the set of strategy profiles of the other players that reach h_i.
- If h_i precedes h_i' ($h_i \rightsquigarrow h_i'$) then $b_i(h_i')$ is derived from $b_i(h_i)$ by Bayes rule whenever possible.

We say that with the belief system b_i and the strategy s_i player i *rational* at the information set $h_i \in H_i$ if there exists no action $a'_{h_i} \in A_{h_i}$ such that only replacing the action $s_i(h_i)$ by a'_{h_i} results in a new strategy s'_i which yields player i a higher expected payoff in T_{h_i} given the belief $b_i(h_i)$ on the other players' strategies $S_{-i}^{T_{h_i}}$.

This notion of rationality takes a local perspective:

- It takes seriously the reasoning about rationality *assuming* that h_i has been reached, whether this assumption is *realistic* or *counterfactual*.
- It considers alternative actions a'_{h_i} only at h_i itself. (Motivation: no way to commit to behavior distinct than the current one.)

One can still prove that our assumptions on the information sets imply dynamic consistency.

Generalizing Extensive-form Rationalizability (Pearce 1984, Battigalli 1997) to Dynamic Unawareness

Inductive definition:

B_i^1 is the set of i's belief systems

$S_i^1 = \{s_i \in S_i$: there exists a belief system $b_i \in B_i^1$ with which for every information set $h_i \in H_i$ player i is rational at $h_i\}$

$\vdots$

$B_i^k = \{b_i \in B_i^{k-1}$: for every information set h_i, if there exists some profile of the other players' strategies $s_{-i} \in S_{-i}^{k-1} = \prod_{j \neq i} S_j^{k-1}$ such that $s_{-i}^{T_{h_i}}$ reaches h_i in the tree T_{h_i}, then $b_i(h_i)$ assigns probability 1 to $S_{-i}^{k-1}\}$

$S_i^k = \{s_i \in S_i$: there exists a belief system $b_i \in B_i^k$ with which for every information set $h_i \in H_i$ player i is rational at $h_i\}$

Generalizing Extensive-form Rationalizability (Pearce 1984, Battigalli 1997) to Dynamic Unawareness

The set of player i's *correlated extensive-form rationalizable strategies* is

$$S_i^{\infty} = \bigcap_{k=1}^{\infty} S_i^k.$$

Remark. $S_i^k \subseteq S_i^{k-1}$ for every $k > 1$.

Proposition 1 *For every finite generalized extensive-form game, the set of extensive-form rationalizable strategies is nonempty.*

Prudent Rationalizability

Let

$$\bar{S}_i^0 = S_i$$

For $k \geq 1$ define inductively

$$\bar{B}_i^k = \left\{ b_i \in B_i : \begin{array}{c} \text{for every information set } h_i, \text{ if there exists some profile} \\ s_{-i} \in \bar{S}_{-i}^{k-1} = \prod_{j \neq i} \bar{S}_j^{k-1} \text{ of the other players' strategies} \\ \text{such that } s_{-i} \text{ reaches } h_i \text{ in the tree } T_{h_i}, \text{ then the support} \\ \text{of } b_i(h_i) \text{ is the set of strategy profiles } s_{-i} \in \bar{S}_{-i}^{k-1,T_{h_i}} \text{ that reach } h_i \end{array} \right\}$$

$$\bar{S}_i^k = \left\{ s_i \in \bar{S}_i^{k-1} : \begin{array}{l} \text{there exists } b_i \in \bar{B}_i^k \text{ such that for all } h_i \in H_i \text{ player } i \\ \text{would be rational at } h_i \end{array} \right\}$$

The set of prudent rationalizable strategies of player i is

$$\bar{S}_i^\infty = \bigcap_{k=1}^{\infty} \bar{S}_i^k$$

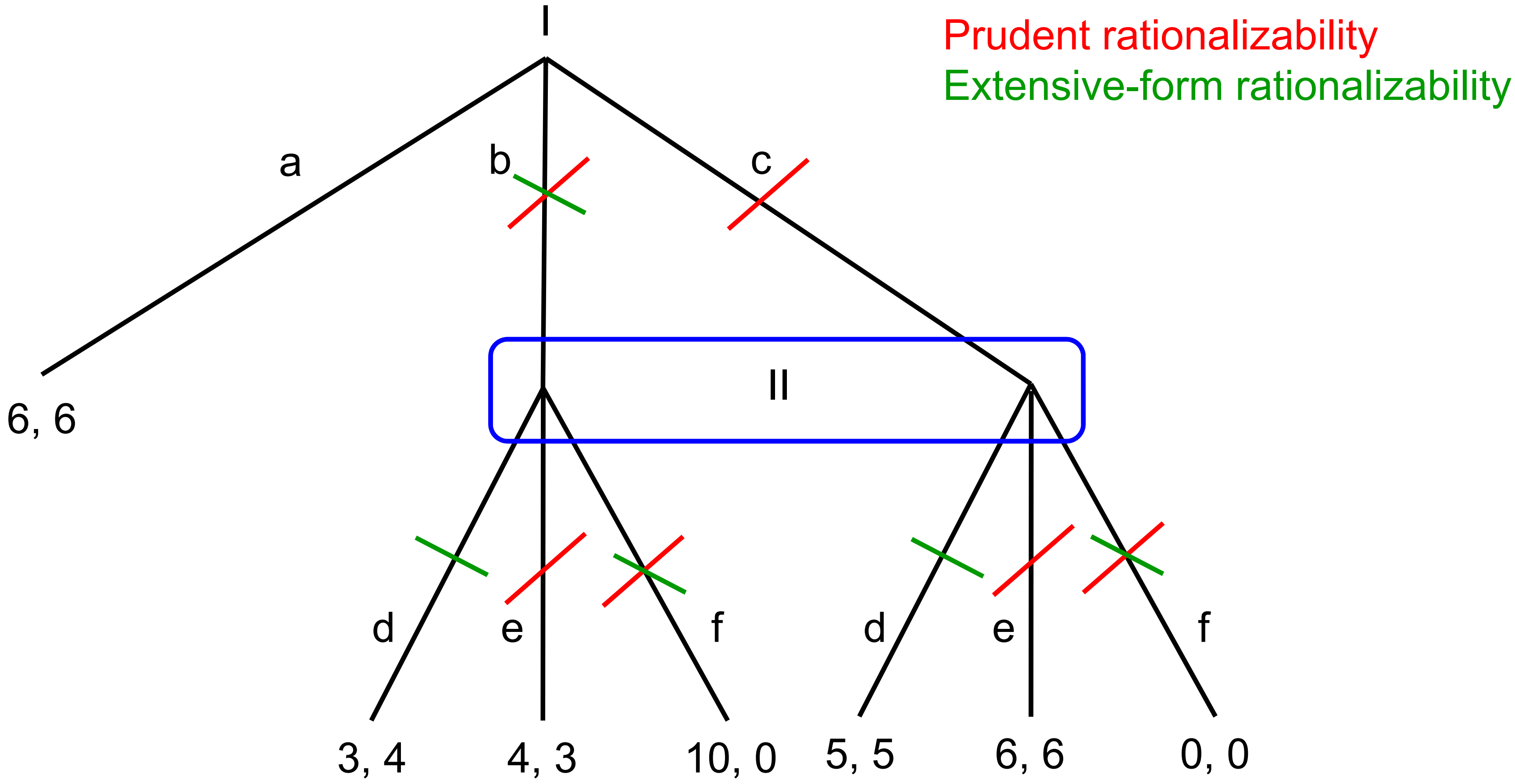
Prudent Rationalizability

Proposition 2 *For every finite generalized extensive-form game, the set of prudent rationalizable strategies is nonempty.*

Conjecture/Proposition: For every finite generalized extensive-form game, the set of outcomes of prudent rationalizable strategies is a subset of outcomes of extensive-form rationalizable stategies.

Prudent rationalizability is not a refinement of extensive-form rationalizability in terms of strategies.

Prudence vs. Rationalizability: Divining the opponent's past behavior



Prudent rationalizability
Extensive-form rationalizability

Application to Verifiable Information

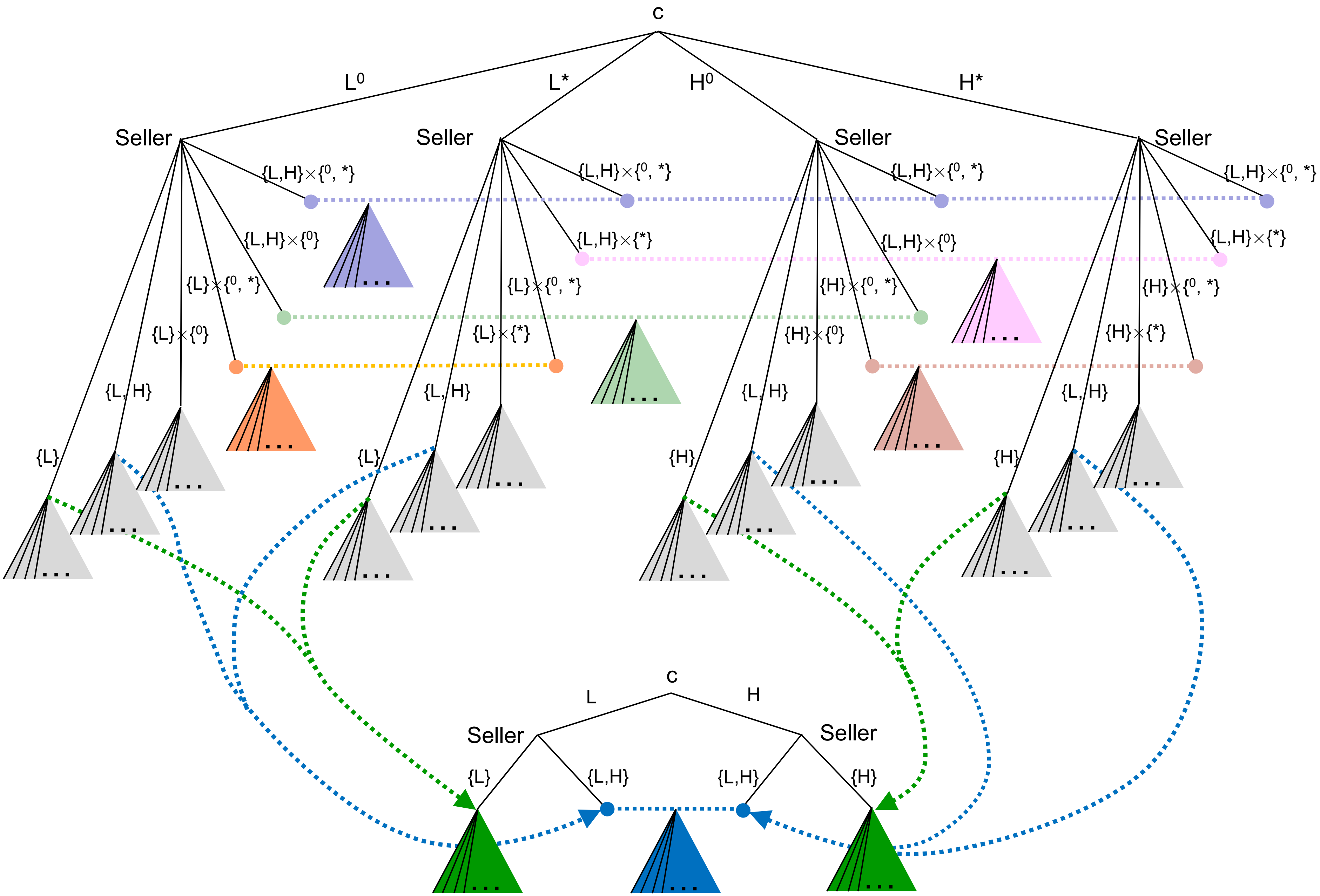
1. With Full Awareness: a la Milgrom and Roberts (1986)

- One seller, one buyer
- One merchandise with quality L or H
- For each quality, the seller is better off from selling a larger quantity.
- The buyer's utility is strictly concave in quantity with single peak $\beta(\cdot)$ and β(L) < β(H).
- Before a sale takes place, the seller can provide a certified signal about the quality: can be imprecise but must be truthful
- The buyer's unique prudent rationalizable strategy is to buy β(L) if presented with {L} or {L, H} and β(H) if presented with {H}, while the seller's unique prudent rationalizable strategy is to certify {L, H} when his quality is L and {H} when is quality is H. **Full unraveling.**

Application to Verifiable Information

2. With Unawareness of a Dimension of Quality

- One merchandise with quality $q \in \{L, H\} \times \{0, *\}$
- For instance, available certificates in state (L,0): $\{L, H\} \times \{0, *\}$, $\{L\} \times \{0, *\}$, $\{L, H\} \times \{0\}$, and $\{L\} \times \{0\}$
- $\beta(L,*) < \beta(L,0) < \beta(H,0) < \beta(H,*)$.
- If the buyer is **fully aware** of both dimensions of quality, we still obtain **full unraveling** by previous arguments.
- Assume now that the buyer is aware only of dimension {L, H} while being unaware of dimension {0, *}. Assume $\beta(L) = \beta(L,0)$ and $\beta(H) = \beta(H,0)$.



c
L⁰
L*
H⁰
H*
Seller
Seller
Seller
Seller
{L,H}×{⁰, *}
{L,H}×{⁰, *}
{L,H}×{⁰, *}
{L,H}×{⁰, *}
{L,H}×{⁰}
{L,H}×{*}
{L,H}×{⁰}
{L,H}×{*}
{L}×{⁰, *}
{L}×{⁰, *}
{H}×{⁰, *}
{H}×{⁰, *}
{L}×{⁰}
{L}×{*}
{H}×{⁰}
{H}×{*}
{L, H}
{L, H}
{L, H}
{L, H}
{L}
{L}
{H}
{H}
c
L
H
Seller
Seller
{L}
{L,H}
{L,H}
{H}

Application to Verifiable Information

2. With Unawareness of a Dimension of Quality

- One merchandise with quality $q \in \{L, H\} \times \{0, *\}$

- For instance, available certificates in state (L,0): $\{L, H\} \times \{0, *\}$, $\{L\} \times \{0, *\}$, $\{L, H\} \times \{0\}$, and $\{L\} \times \{0\}$

- $\beta(L,*) < \beta(L,0) < \beta(H,0) < \beta(H,*)$.

- If the buyer is **fully aware** of both dimensions of quality, we still obtain **full unraveling** by previous arguments.

- Assume now that the buyer is aware only of dimension {L, H} while being unaware of dimension {0, *}. Assume $\beta(L) = \beta(L,0)$ and $\beta(H,0) = \beta(H)$.

- In the verifiable information model in which the buyer is unaware of some dimension of the the good's quality, the seller does **not fully reveal** the quality in any prudent rationalizable outcome. E.g., (L, *).

Conditional Dominance

Can we capture these solution concepts in “normal-form“ games with unawareness?

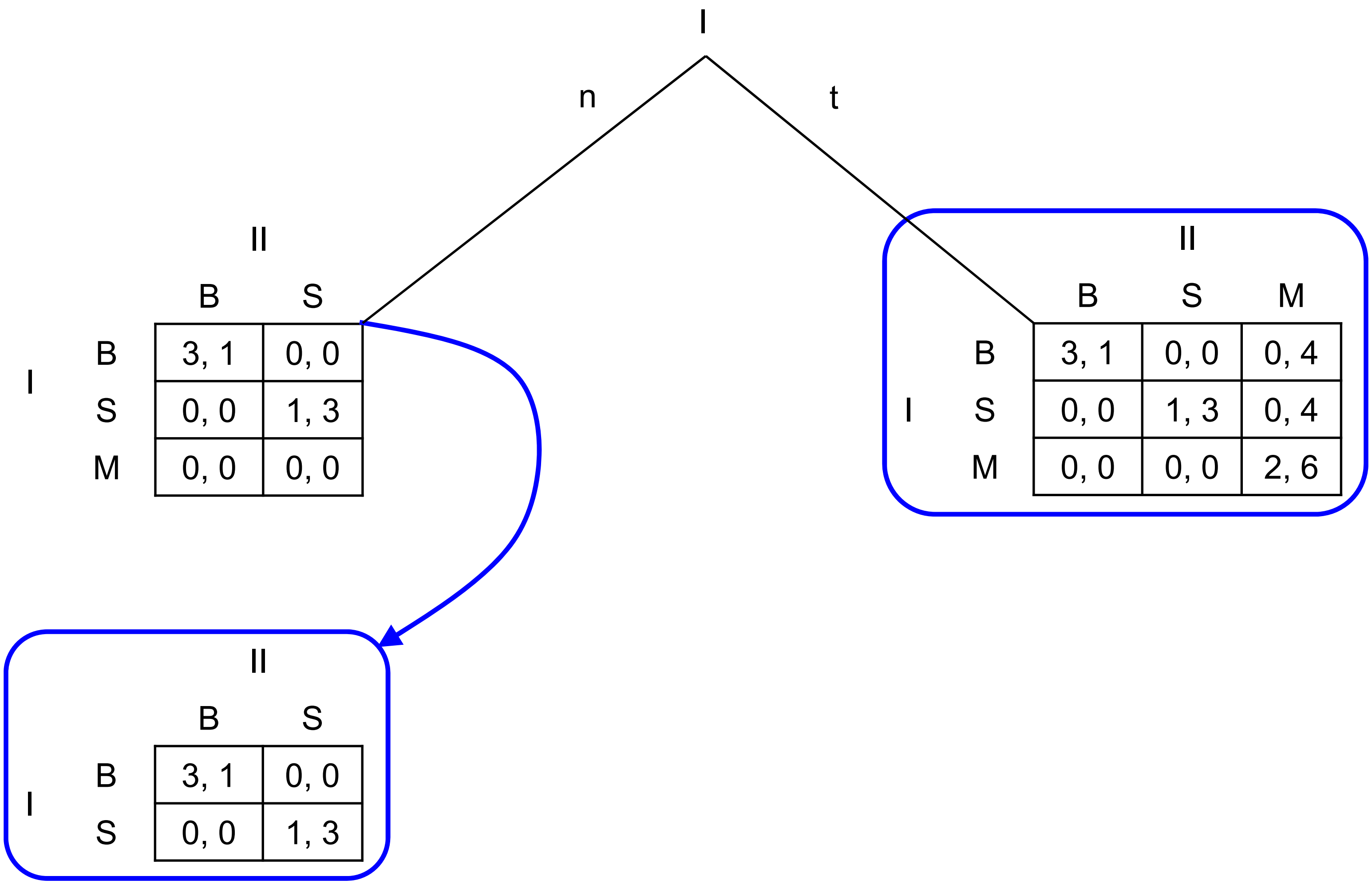
How crucial is the “time-structure” for the analysis of strategic interaction under unawareness?

For standard extensive-form games, Shimoji and Watson (JET 1998) show that extensive-form rationalizability is characterized by iterated elimination of conditionally strictly dominated strategies.

Chen and Micali (TE 2012) show order-independence of iterated elimination of conditionally strictly dominated strategies.

How about dynamic games with unawareness?

Example: A kind of “Battle of the Sexes”

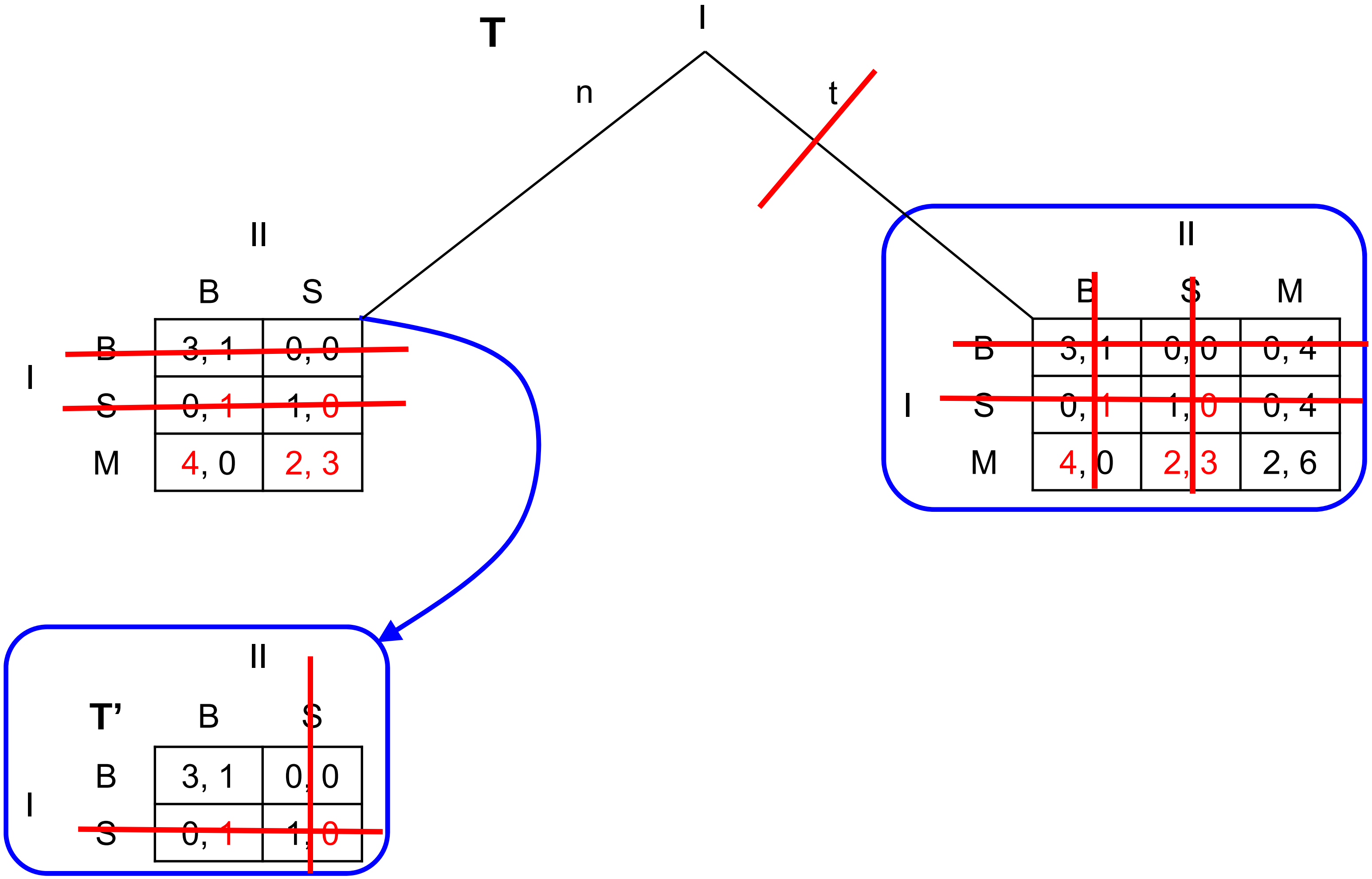


I	B	S
B	3, 1	0, 0
S	0, 0	1, 3
M	0, 0	0, 0

I	B	S	M
B	3, 1	0, 0	0, 4
S	0, 0	1, 3	0, 4
M	0, 0	0, 0	2, 6

I	B	S
B	3, 1	0, 0
S	0, 0	1, 3

Example: ~~A kind of "Battle of the Sexes"~~



T, after n:

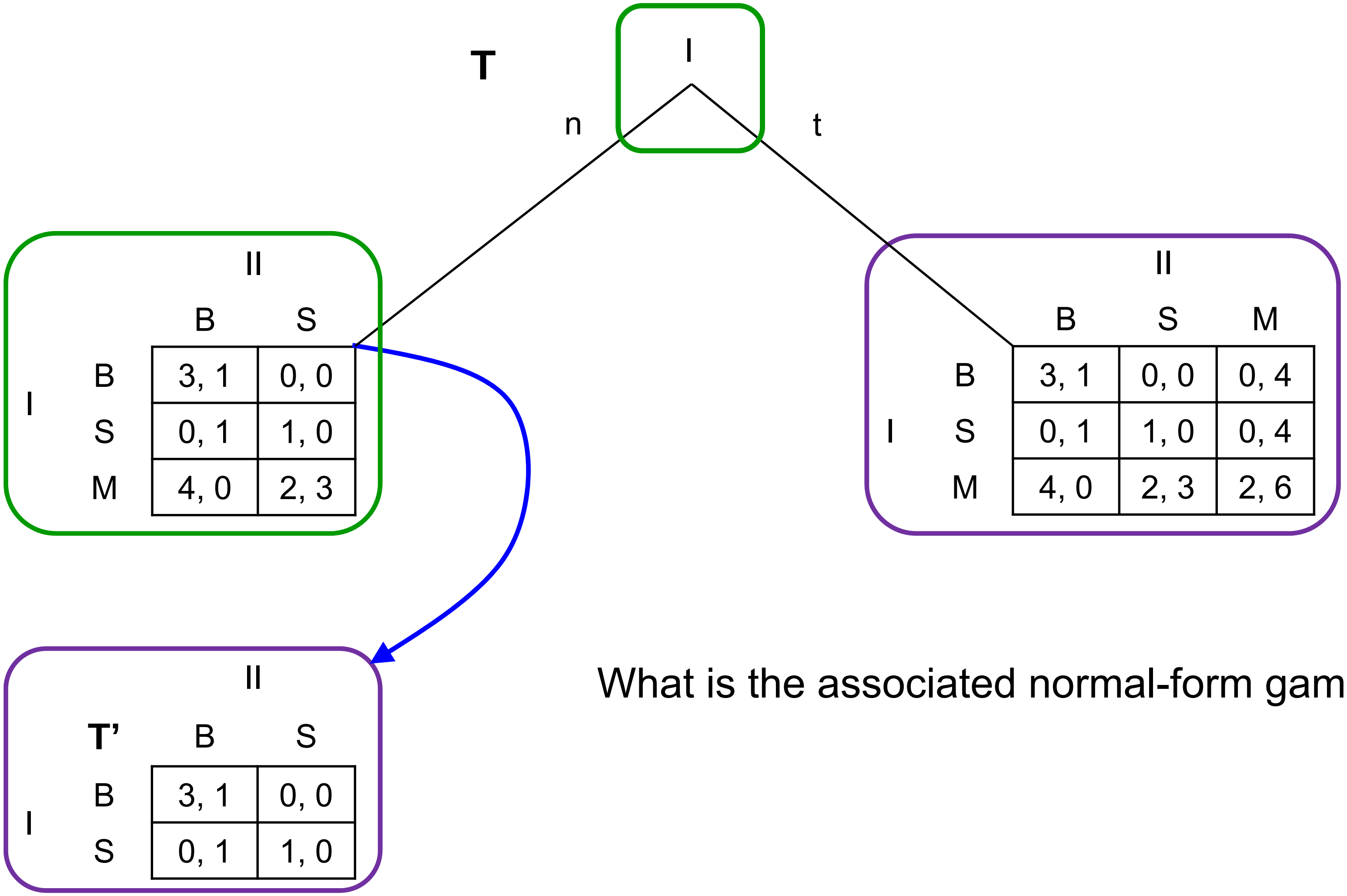
		II: B	II: S
I	~~B~~	~~3, 1~~	~~0, 0~~
	~~S~~	~~0, 1~~	~~1, 0~~
	M	4, 0	2, 3

After ~~t~~:

		II: ~~B~~	II: ~~S~~	II: M
I	~~B~~	~~3, 1~~	~~0, 0~~	~~0, 4~~
	~~S~~	~~0, 1~~	~~1, 0~~	~~0, 4~~
	M	~~4, 0~~	~~2, 3~~	2, 6

T'

		II: B	II: ~~S~~
I	B	3, 1	~~0, 0~~
	~~S~~	~~0, 1~~	~~1, 0~~

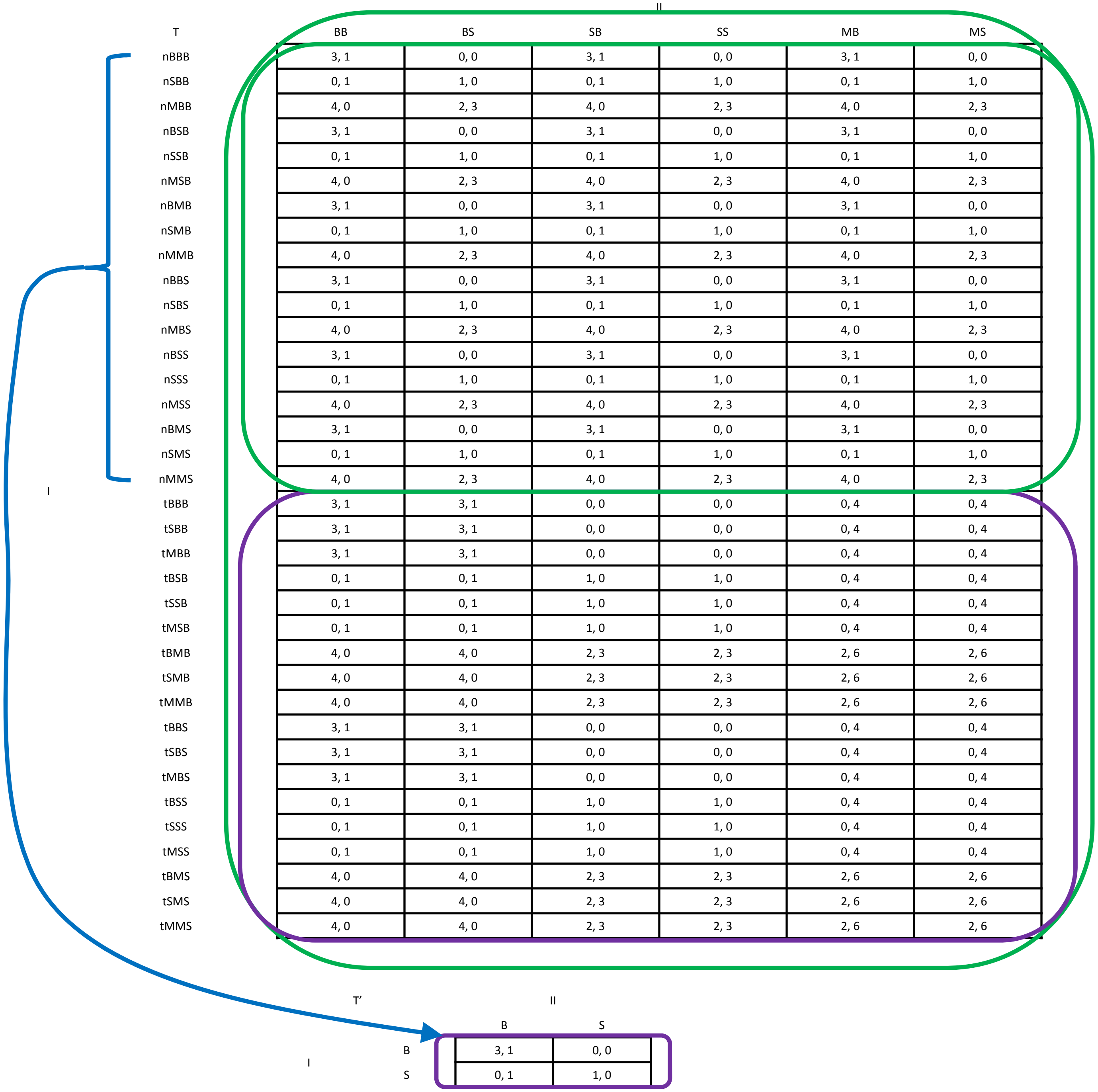


	B	S
B	3, 1	0, 0
S	0, 1	1, 0
M	4, 0	2, 3

	B	S	M
B	3, 1	0, 0	0, 4
S	0, 1	1, 0	0, 4
M	4, 0	2, 3	2, 6

T'	B	S
B	3, 1	0, 0
S	0, 1	1, 0

What is the associated normal-form game?



II

T	BB	BS	SB	SS	MB	MS
nBBB	3, 1	0, 0	3, 1	0, 0	3, 1	0, 0
nSBB	0, 1	1, 0	0, 1	1, 0	0, 1	1, 0
nMBB	4, 0	2, 3	4, 0	2, 3	4, 0	2, 3
nBSB	3, 1	0, 0	3, 1	0, 0	3, 1	0, 0
nSSB	0, 1	1, 0	0, 1	1, 0	0, 1	1, 0
nMSB	4, 0	2, 3	4, 0	2, 3	4, 0	2, 3
nBMB	3, 1	0, 0	3, 1	0, 0	3, 1	0, 0
nSMB	0, 1	1, 0	0, 1	1, 0	0, 1	1, 0
nMMB	4, 0	2, 3	4, 0	2, 3	4, 0	2, 3
nBBS	3, 1	0, 0	3, 1	0, 0	3, 1	0, 0
nSBS	0, 1	1, 0	0, 1	1, 0	0, 1	1, 0
nMBS	4, 0	2, 3	4, 0	2, 3	4, 0	2, 3
nBSS	3, 1	0, 0	3, 1	0, 0	3, 1	0, 0
nSSS	0, 1	1, 0	0, 1	1, 0	0, 1	1, 0
nMSS	4, 0	2, 3	4, 0	2, 3	4, 0	2, 3
nBMS	3, 1	0, 0	3, 1	0, 0	3, 1	0, 0
nSMS	0, 1	1, 0	0, 1	1, 0	0, 1	1, 0
nMMS	4, 0	2, 3	4, 0	2, 3	4, 0	2, 3
tBBB	3, 1	3, 1	0, 0	0, 0	0, 4	0, 4
tSBB	3, 1	3, 1	0, 0	0, 0	0, 4	0, 4
tMBB	3, 1	3, 1	0, 0	0, 0	0, 4	0, 4
tBSB	0, 1	0, 1	1, 0	1, 0	0, 4	0, 4
tSSB	0, 1	0, 1	1, 0	1, 0	0, 4	0, 4
tMSB	0, 1	0, 1	1, 0	1, 0	0, 4	0, 4
tBMB	4, 0	4, 0	2, 3	2, 3	2, 6	2, 6
tSMB	4, 0	4, 0	2, 3	2, 3	2, 6	2, 6
tMMB	4, 0	4, 0	2, 3	2, 3	2, 6	2, 6
tBBS	3, 1	3, 1	0, 0	0, 0	0, 4	0, 4
tSBS	3, 1	3, 1	0, 0	0, 0	0, 4	0, 4
tMBS	3, 1	3, 1	0, 0	0, 0	0, 4	0, 4
tBSS	0, 1	0, 1	1, 0	1, 0	0, 4	0, 4
tSSS	0, 1	0, 1	1, 0	1, 0	0, 4	0, 4
tMSS	0, 1	0, 1	1, 0	1, 0	0, 4	0, 4
tBMS	4, 0	4, 0	2, 3	2, 3	2, 6	2, 6
tSMS	4, 0	4, 0	2, 3	2, 3	2, 6	2, 6
tMMS	4, 0	4, 0	2, 3	2, 3	2, 6	2, 6

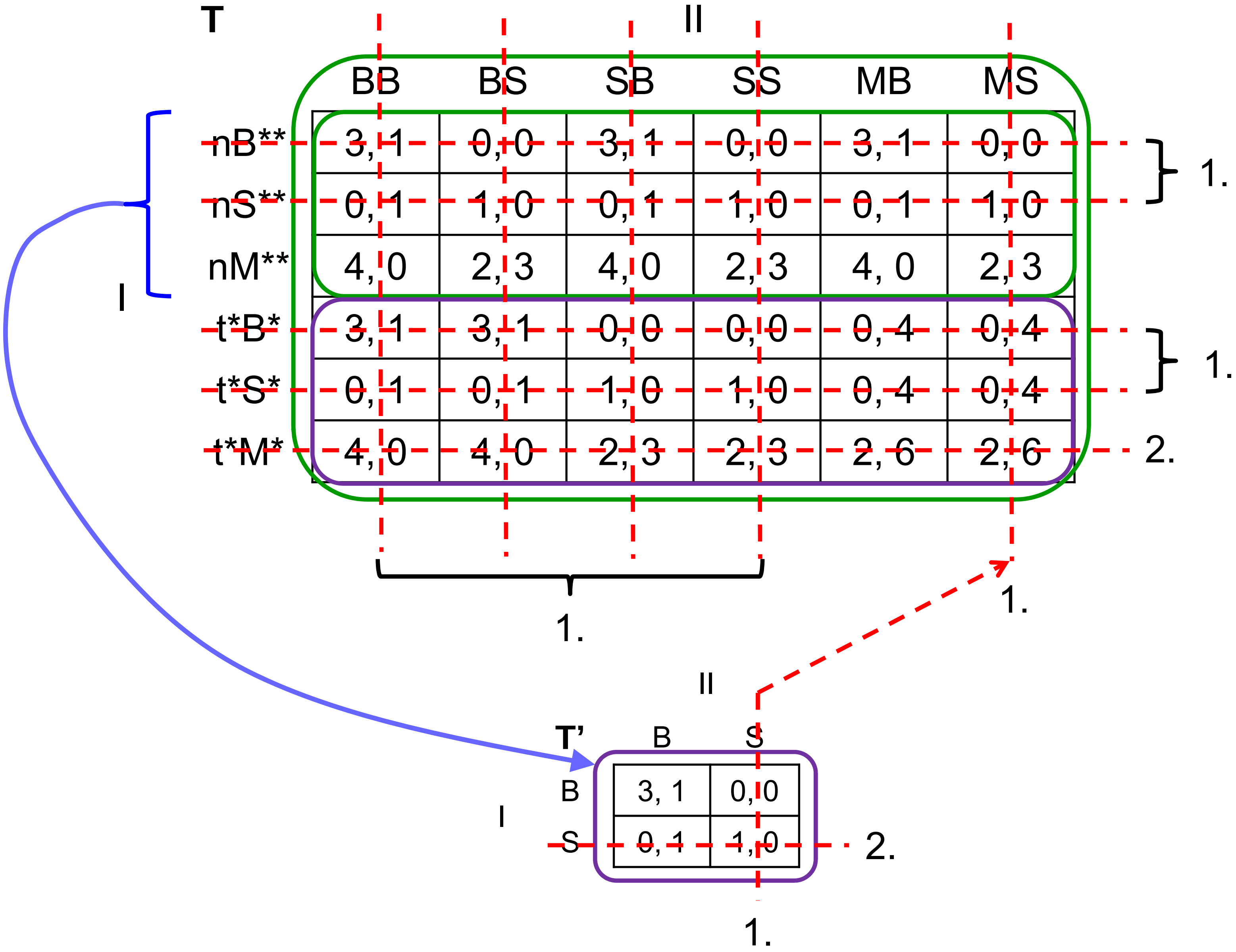
I

T’

II

	B	S
B	3, 1	0, 0
S	0, 1	1, 0

I



	BB	BS	SB	SS	MB	MS
nB**	3, 1	0, 0	3, 1	0, 0	3, 1	0, 0
nS**	0, 1	1, 0	0, 1	1, 0	0, 1	1, 0
nM**	4, 0	2, 3	4, 0	2, 3	4, 0	2, 3
t*B*	3, 1	3, 1	0, 0	0, 0	0, 4	0, 4
t*S*	0, 1	0, 1	1, 0	1, 0	0, 4	0, 4
t*M*	4, 0	4, 0	2, 3	2, 3	2, 6	2, 6

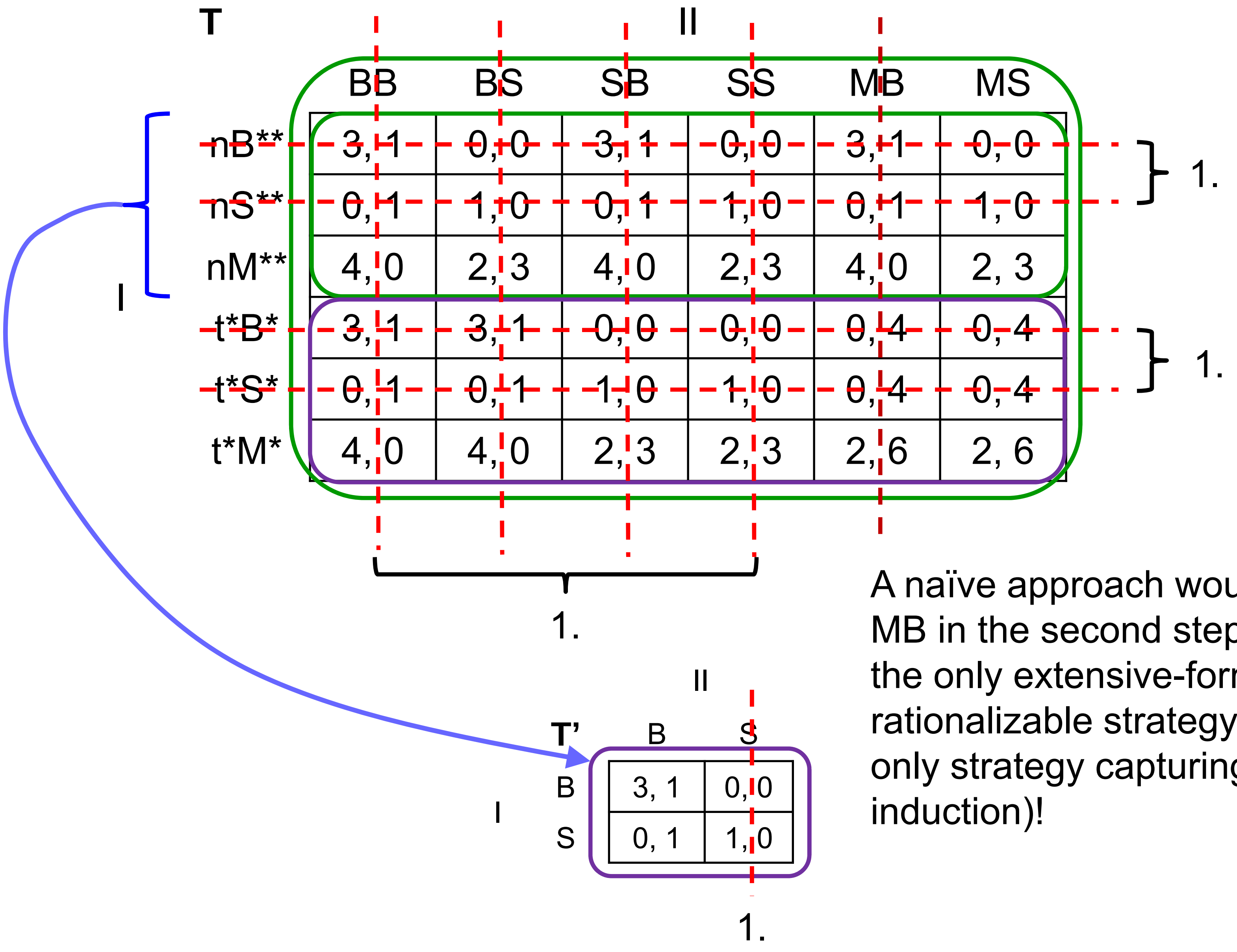
	B	S
B	3, 1	0, 0
S	0, 1	1, 0

Iterated conditional dominance coincides with extensive-form rationalizability.

Iterated conditional dominance requires normal-form information sets and thus structure from the extensive-form. Can we dispense with any extensive-form structure entirely and obtain forward-induction outcomes in the associated normal-form game?

=> Iterated admissibility?

Iterated admissibility:

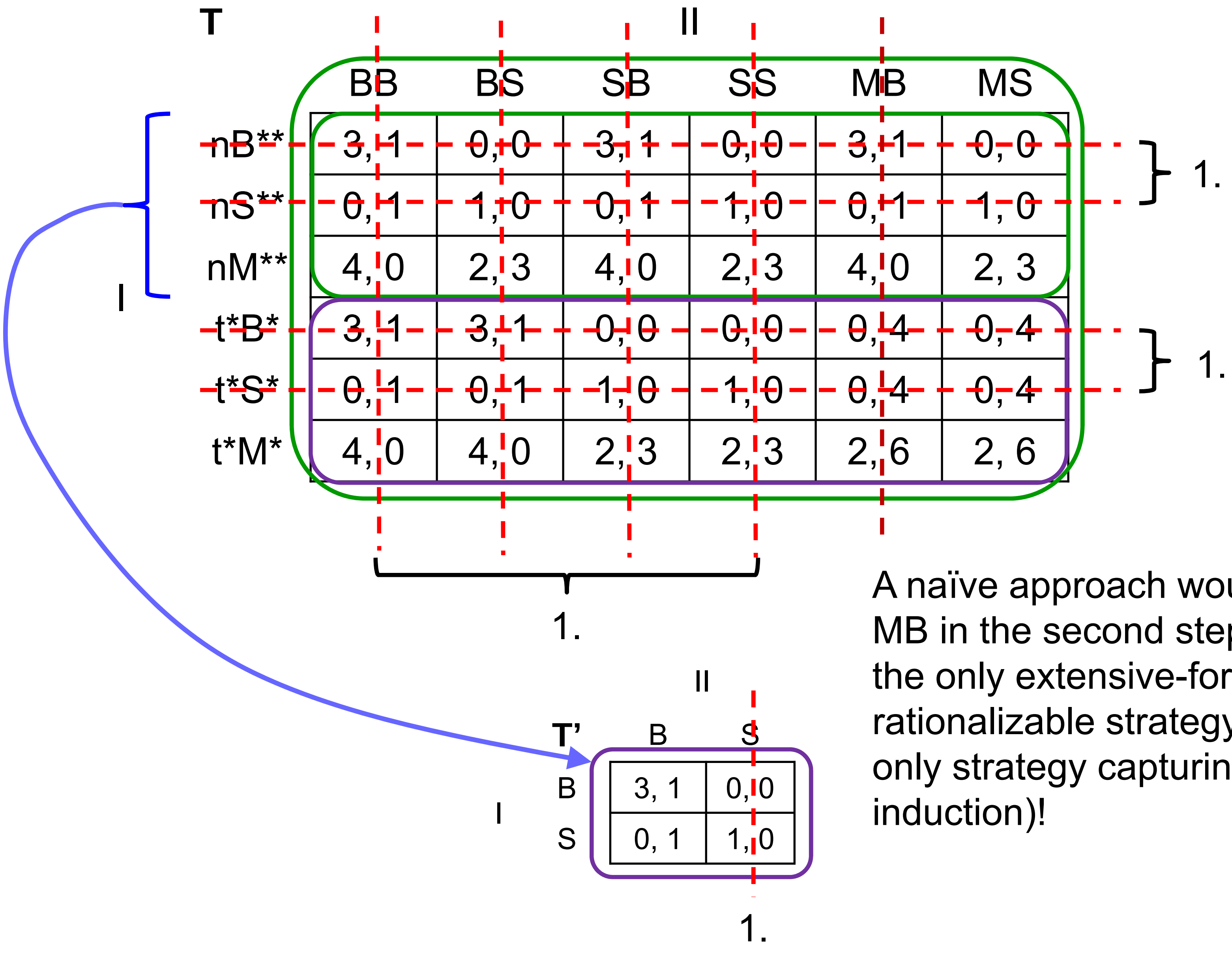


	BB	BS	SB	SS	MB	MS
nB**	3, 1	0, 0	3, 1	0, 0	3, 1	0, 0
nS**	0, 1	1, 0	0, 1	1, 0	0, 1	1, 0
nM**	4, 0	2, 3	4, 0	2, 3	4, 0	2, 3
t*B*	3, 1	3, 1	0, 0	0, 0	0, 4	0, 4
t*S*	0, 1	0, 1	1, 0	1, 0	0, 4	0, 4
t*M*	4, 0	4, 0	2, 3	2, 3	2, 6	2, 6

	B	S
B	3, 1	0, 0
S	0, 1	1, 0

A naïve approach would delete MB in the second step, which is the only extensive-form rationalizable strategy (i.e., the only strategy capturing forward induction)!

How about iterated admissibility?



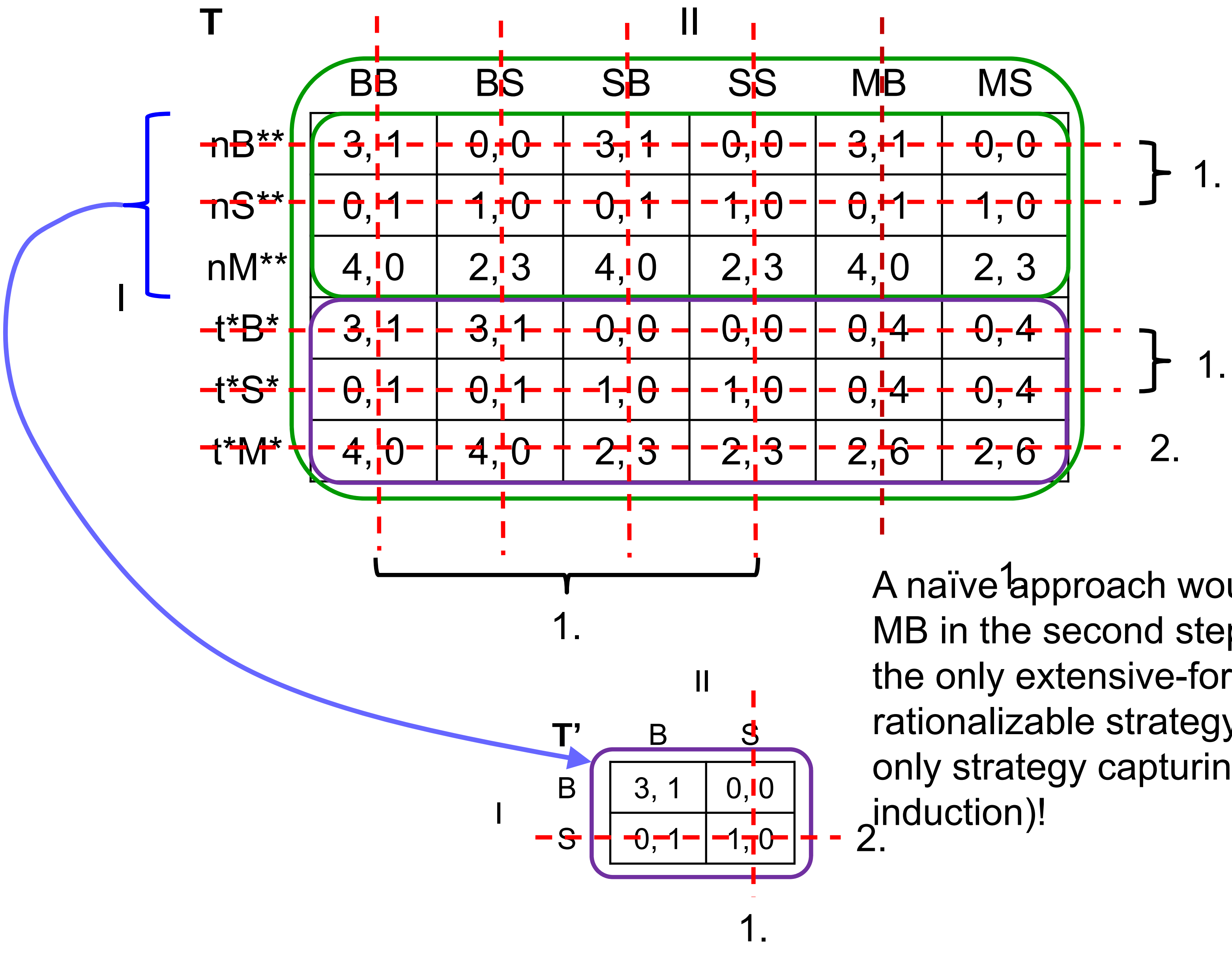
	BB	BS	SB	SS	MB	MS
nB**	3, 1	0, 0	3, 1	0, 0	3, 1	0, 0
nS**	0, 1	1, 0	0, 1	1, 0	0, 1	1, 0
nM**	4, 0	2, 3	4, 0	2, 3	4, 0	2, 3
t*B*	3, 1	3, 1	0, 0	0, 0	0, 4	0, 4
t*S*	0, 1	0, 1	1, 0	1, 0	0, 4	0, 4
t*M*	4, 0	4, 0	2, 3	2, 3	2, 6	2, 6

	B	S
B	3, 1	0, 0
S	0, 1	1, 0

A naïve approach would delete MB in the second step, which is the only extensive-form rationalizable strategy (i.e., the only strategy capturing forward induction)!

Iterated admissibility must rely on information sets (i.e., extensive-form structures).

How about iterated admissibility?



	BB	BS	SB	SS	MB	MS
nB**	3, 1	0, 0	3, 1	0, 0	3, 1	0, 0
nS**	0, 1	1, 0	0, 1	1, 0	0, 1	1, 0
nM**	4, 0	2, 3	4, 0	2, 3	4, 0	2, 3
t*B*	3, 1	3, 1	0, 0	0, 0	0, 4	0, 4
t*S*	0, 1	0, 1	1, 0	1, 0	0, 4	0, 4
t*M*	4, 0	4, 0	2, 3	2, 3	2, 6	2, 6

	B	S
B	3, 1	0, 0
S	0, 1	1, 0

A naïve approach would delete MB in the second step, which is the only extensive-form rationalizable strategy (i.e., the only strategy capturing forward induction)!

Iterated admissibility must rely on information sets (i.e., extensive-form structures).

We defined the normal form associated to dynamic games with unawareness.

We characterize extensive-form rationalizability by iterative conditional strict dominance and prudent rationalizability by iterative conditional weak dominance.

We show that iterative admissibility in the associated normal form must depend on information sets.

We show that iterative conditional weak dominance coincides with iterated admissibility in dynamic games with unawareness.

Other approaches to extensive-form games with unawareness

An Example: Unforeseen Roadwork (Feinberg 2012)

Alice (baker), Bob (coffee-shop owner), Carol (Alice's worker)

Alice has to deliver pastries to Bob on Monday or with penalties on Tuesdays or Wednesdays unless unforeseen contingencies occur

Bob notices unforeseen roadwork on Monday morning that is likely to delay delivery; Bob thinks that Alice is unaware of roadwork

Bob can decide to call Alice and renegotiate later delivery

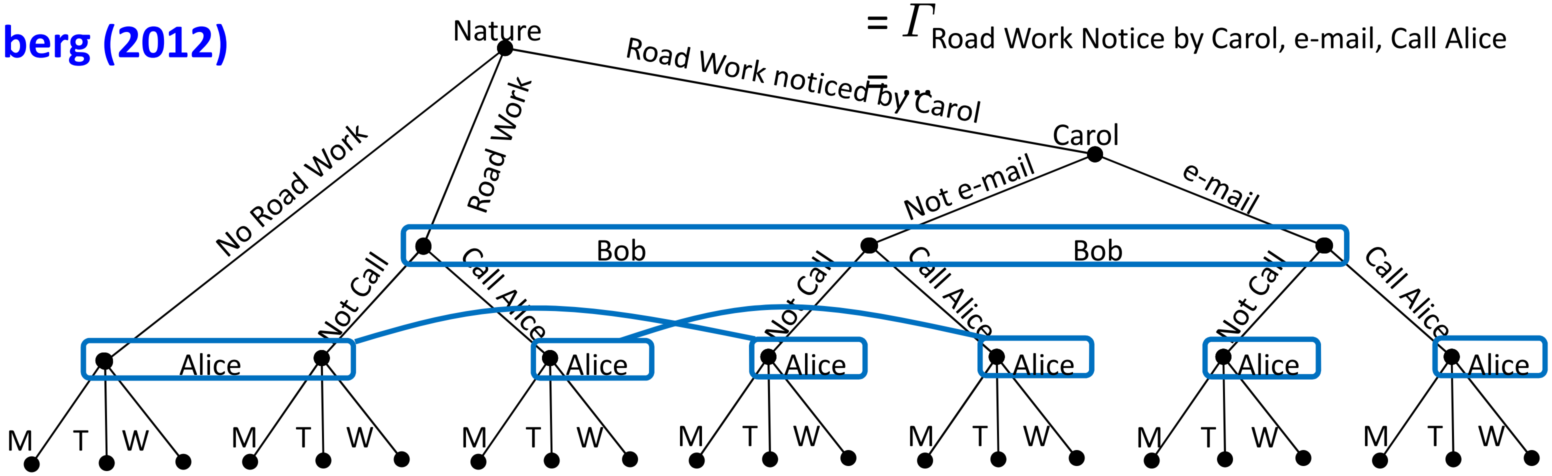
Carol notices upcoming roadwork on Sunday and emails Alice; Bob is unaware of Carol

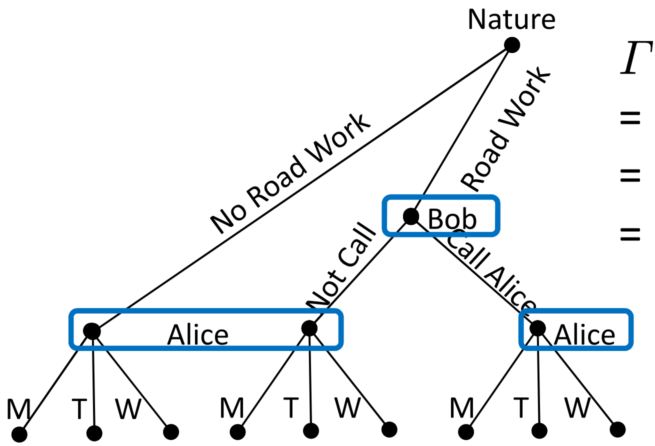
Feinberg (2012)

$\Gamma_{\emptyset} = \Gamma_{\text{Road Work noticed by Carol, e-mail, Not Call}}$

$= \Gamma_{\text{Road Work Notice by Carol, e-mail, Call Alice}}$

$= \ldots$



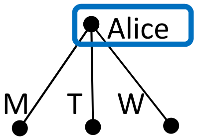


$\Gamma_{\text{Road Work}} = \Gamma_{\text{Road Work noticed by Carol, No e-mail,}}$

$= \Gamma_{\text{Road Work noticed by Carol, E-mail}} = \Gamma_{\text{Road Work, Call Alice}}$

$= \Gamma_{\text{Road Work noticed by Carol, No e-mail, Call Alice}}$

$= \ldots$



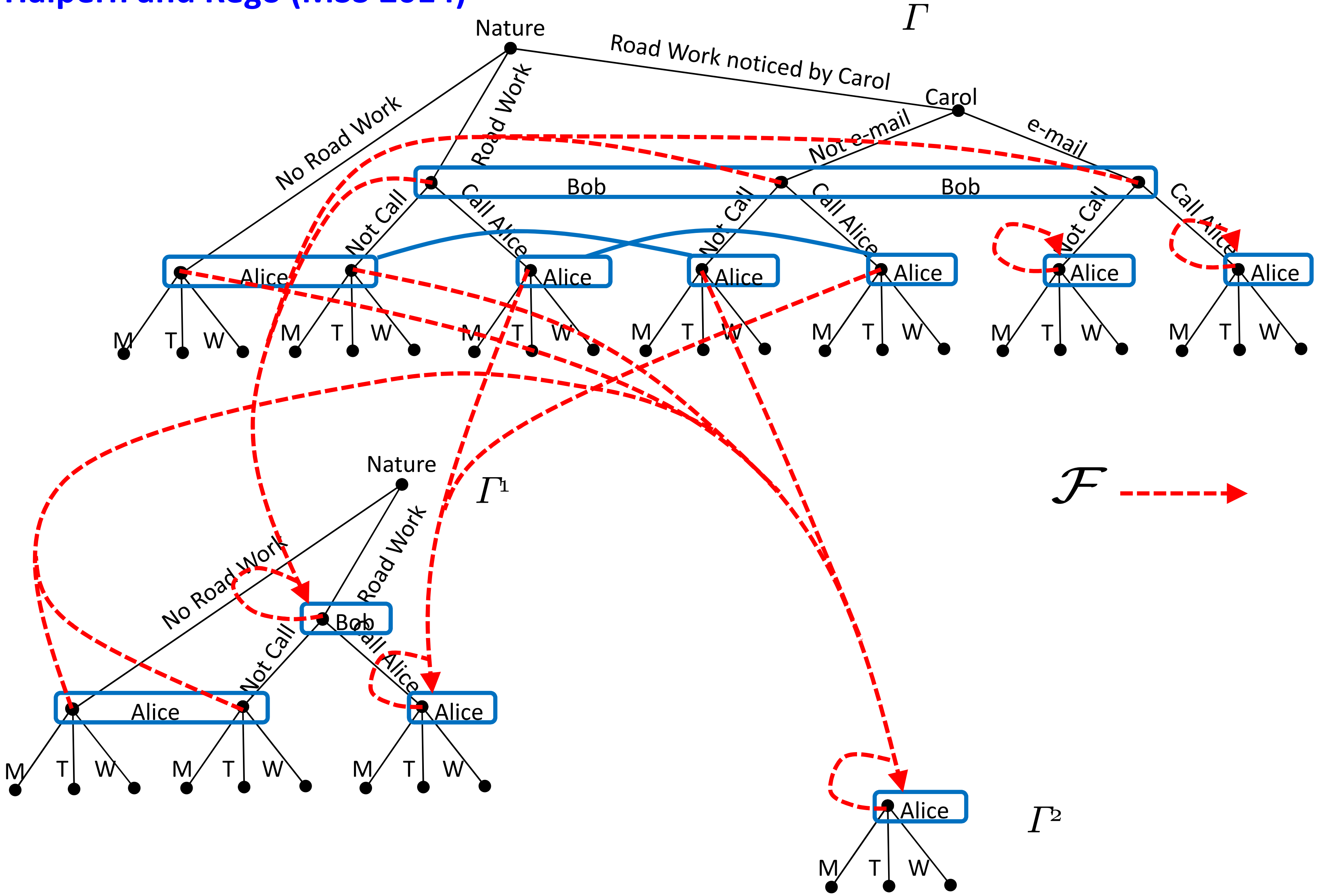
$\Gamma_{\text{No Road Work}}$

$= \Gamma_{\text{Road Work, Not Call}}$

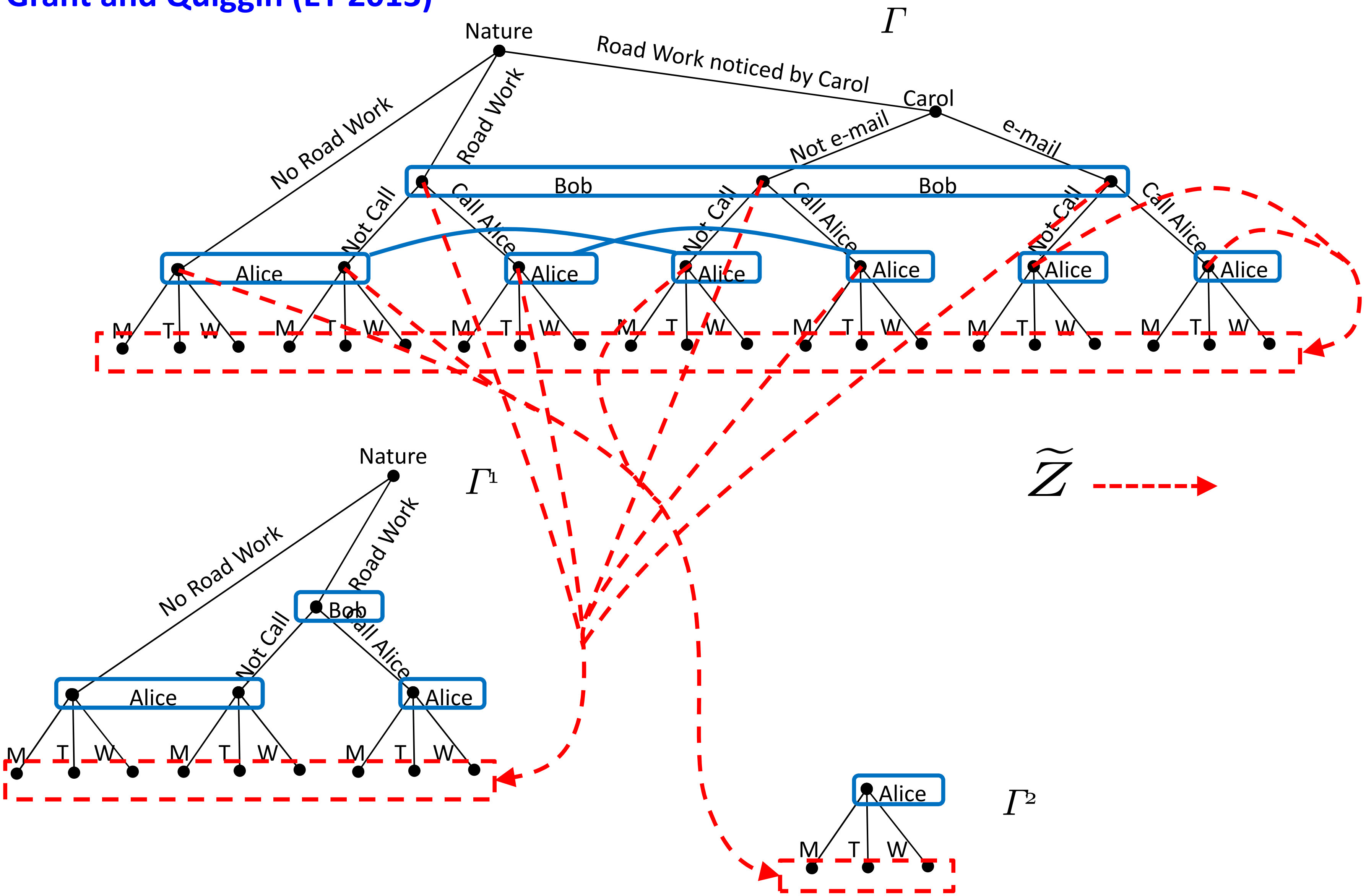
$= \Gamma_{\text{Road Work noticed by Carol, No e-mail, Not Call}}$

$= \ldots$

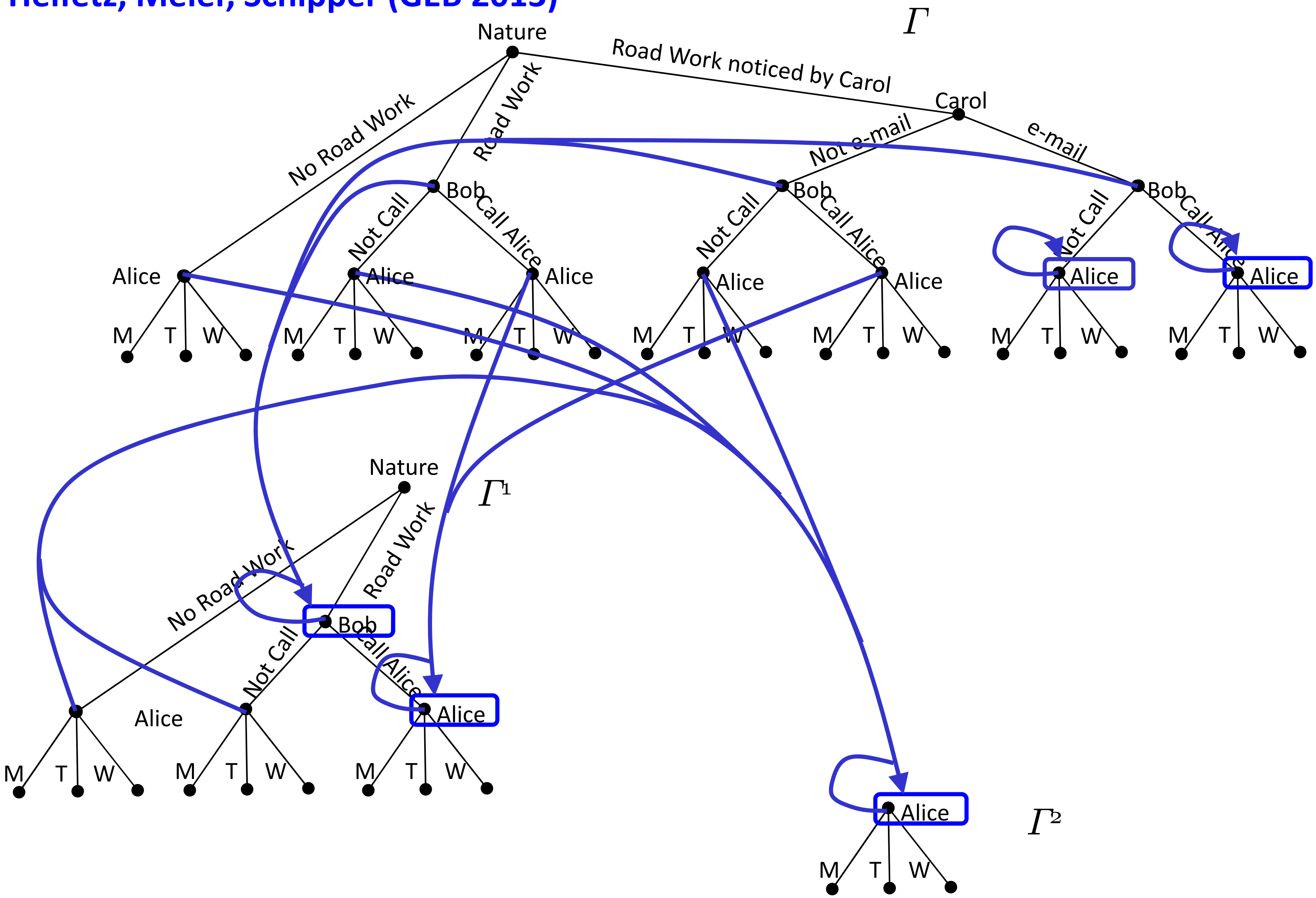
Halpern and Rego (MSS 2014)



Grant and Quiggin (ET 2013)



Heifetz, Meier, Schipper (GEB 2013)



In a Nutshell:

Similarity: Every approach uses a forest of game trees.

Main differences: How is dynamic awareness represented in the forest of game trees:

- Feinberg: infinite sequences of views
- Halpern-Rego: awareness correspondence
- Grant-Quggin: perception correspondence
- Heifetz, Meier, and Schipper: information sets

Solution concepts:

- Feinberg, Halpern-Rego, Grant-Quiggin: Sequential equilibrium
- Heifetz, Meier, and Schipper: Extensive-form rationalizability

Self-confirming games

Self-confirming games

Questions:

- How do we arrive at our perceptions of strategic contexts?
- How to model the process of discovering novel features of "repeated" strategic contexts?
- Is there something like a "steady-state" of perceptions and if there is do they necessarily involve a common perception among players?
- Is there a notion of equilibrium that makes sense in dynamic strategic interaction under unawareness?

Equilibrium under unawareness?

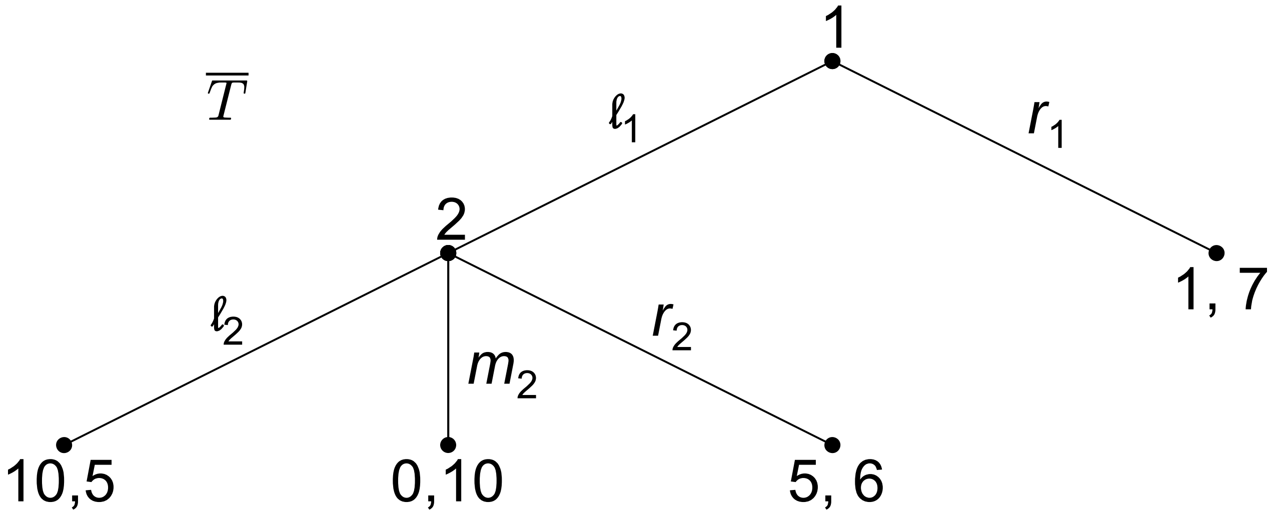
Problems:

- Although mathematical definitions of standard equilibrium concepts can be extended to games with unawareness (Halpern and Rego, 2014, Rego and Halpern, 2012, Feinberg, 2012, Li 2008, Grant and Quiggin, 2013, Ozbay, 2007, Meier and Schipper, 2013), they cannot be interpreted as steady-states of behavior.
- Players could not learn equilibrium through repeated play because they become aware of novel features. The perception of the game may "change" between "repetitions" and even along the supposed equilibrium path of a given game with unawareness.
- How to reconcile equilibrium notions and non-equilibrium solutions (i.e., extensive-form rationalizability) in games with unawareness?

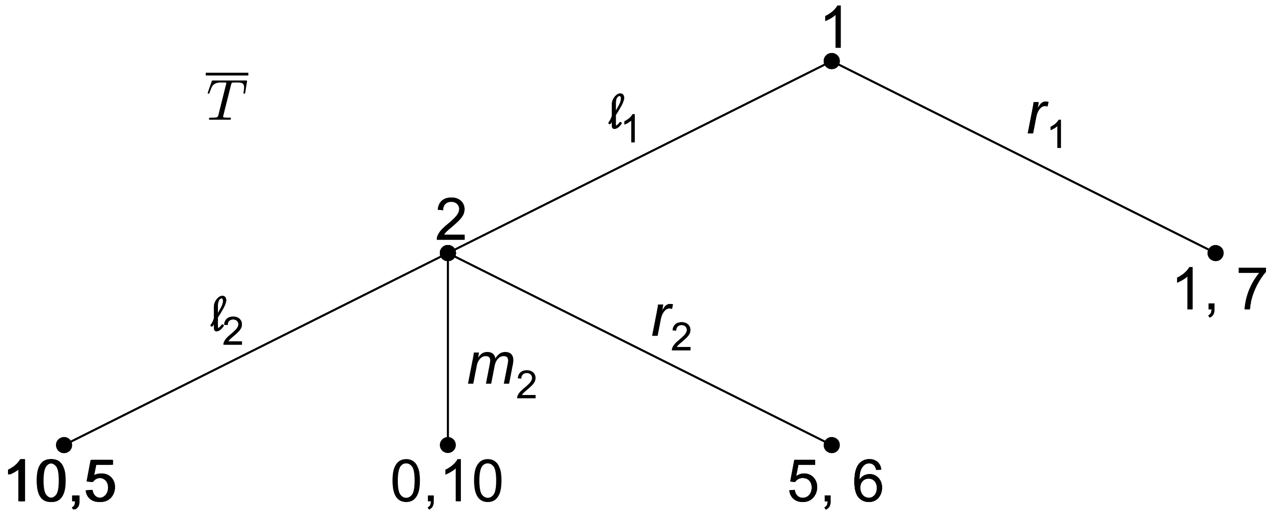
Main ideas

- Formal model of rationalizable discoveries in games with unawareness
- Explain endogenously how players perceive strategic contexts.
- Existence of equilibrium of perceptions and behavior: Show that for every game with unawareness there exists a discovered version that can be viewed as a steady-state of player's perceptions and possess an equilibrium that can be interpreted as a steady-state of behavior.

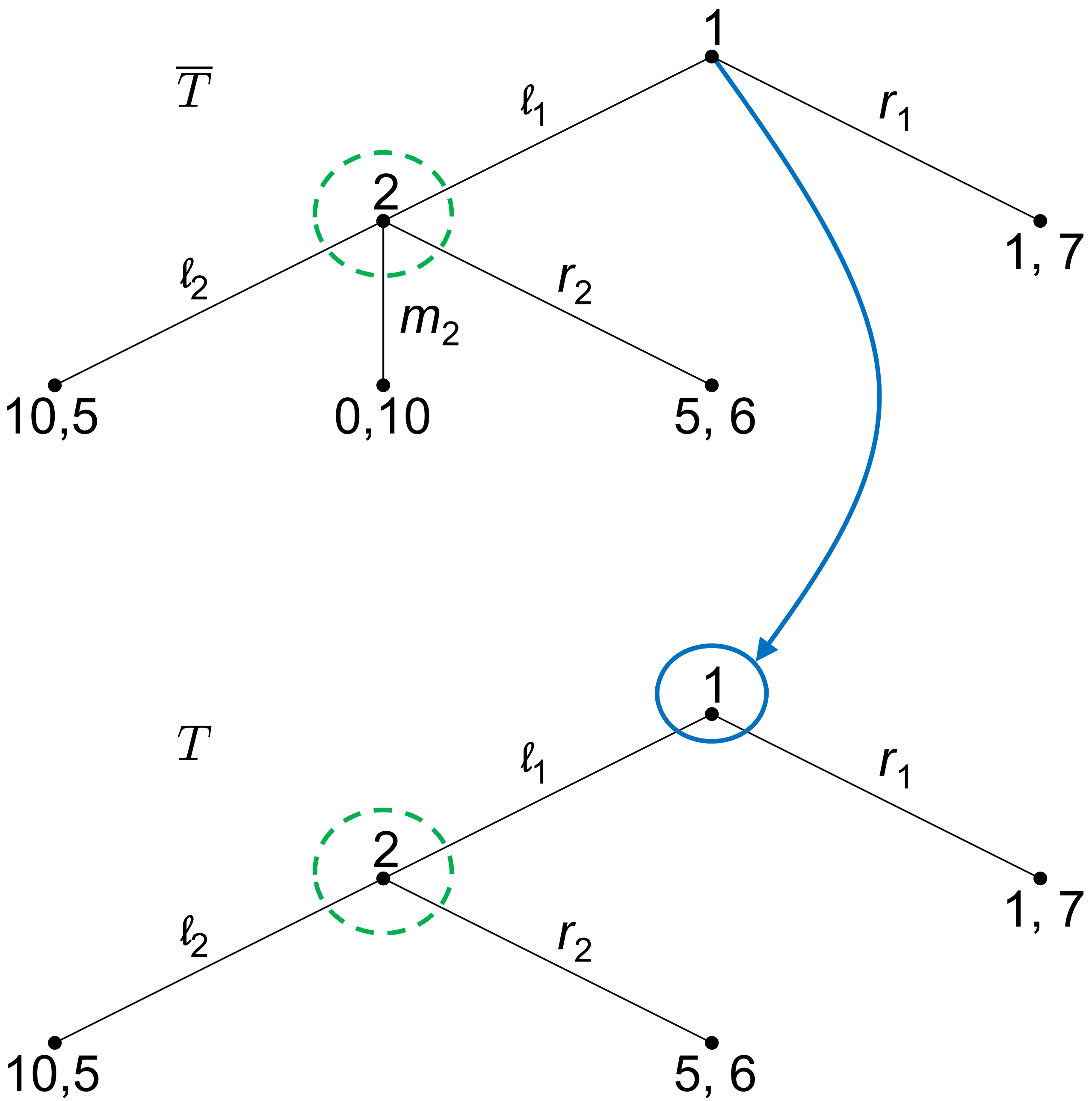
Examples

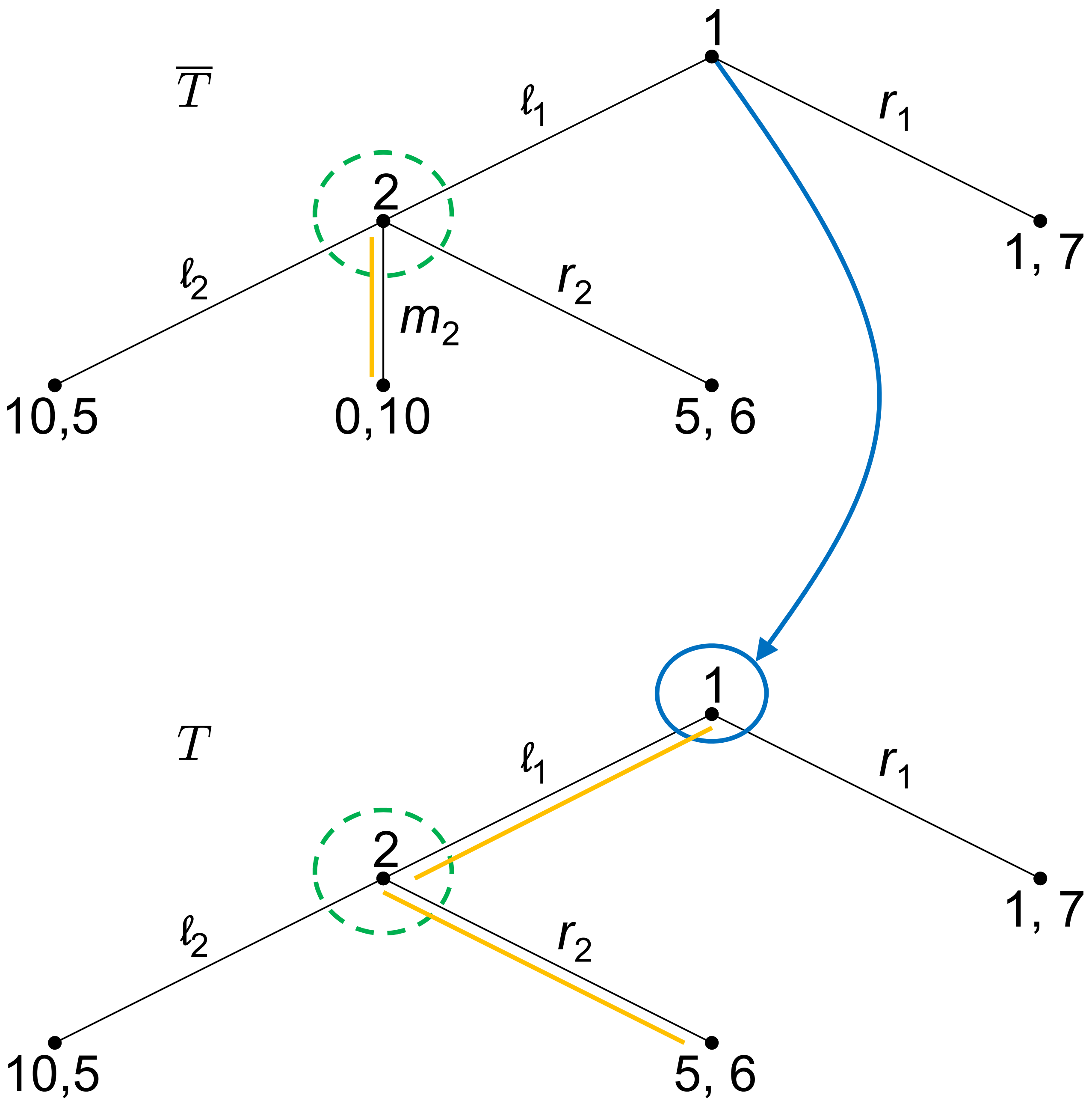

$\overline{T}$
1
ℓ_1
r_1
2
1, 7
ℓ_2
m_2
r_2
10,5
0,10
5, 6

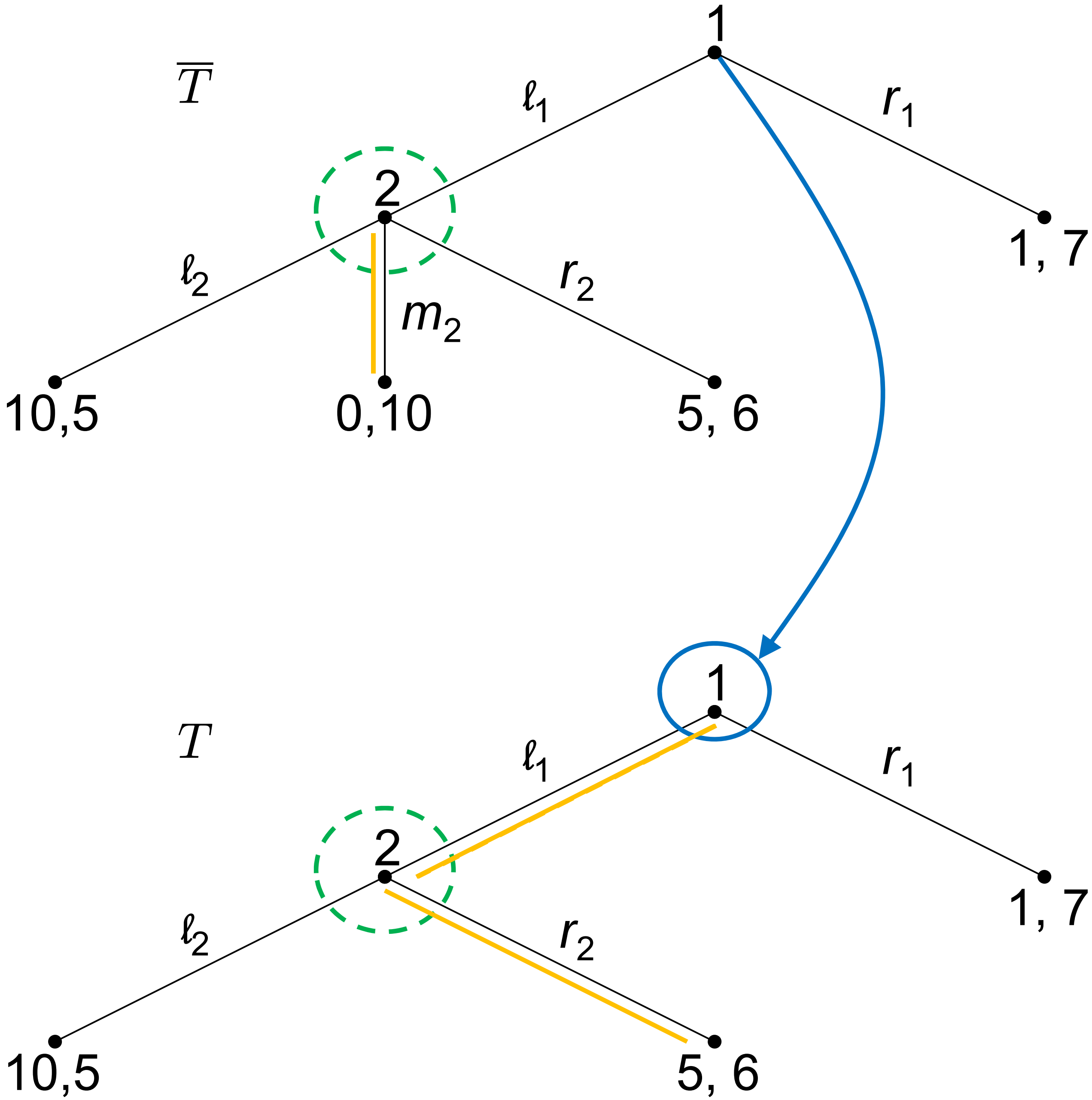
$\overline{T}$

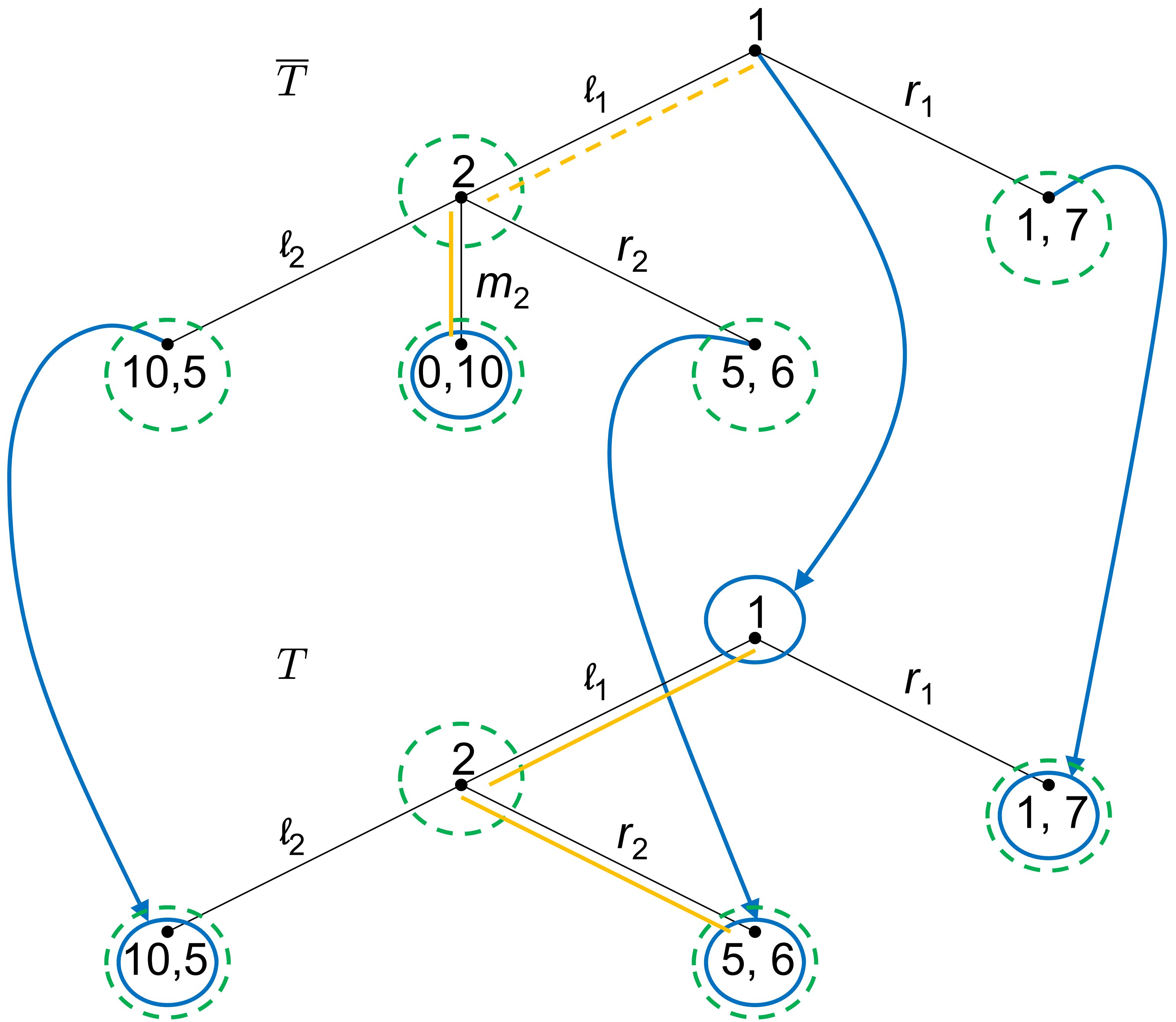


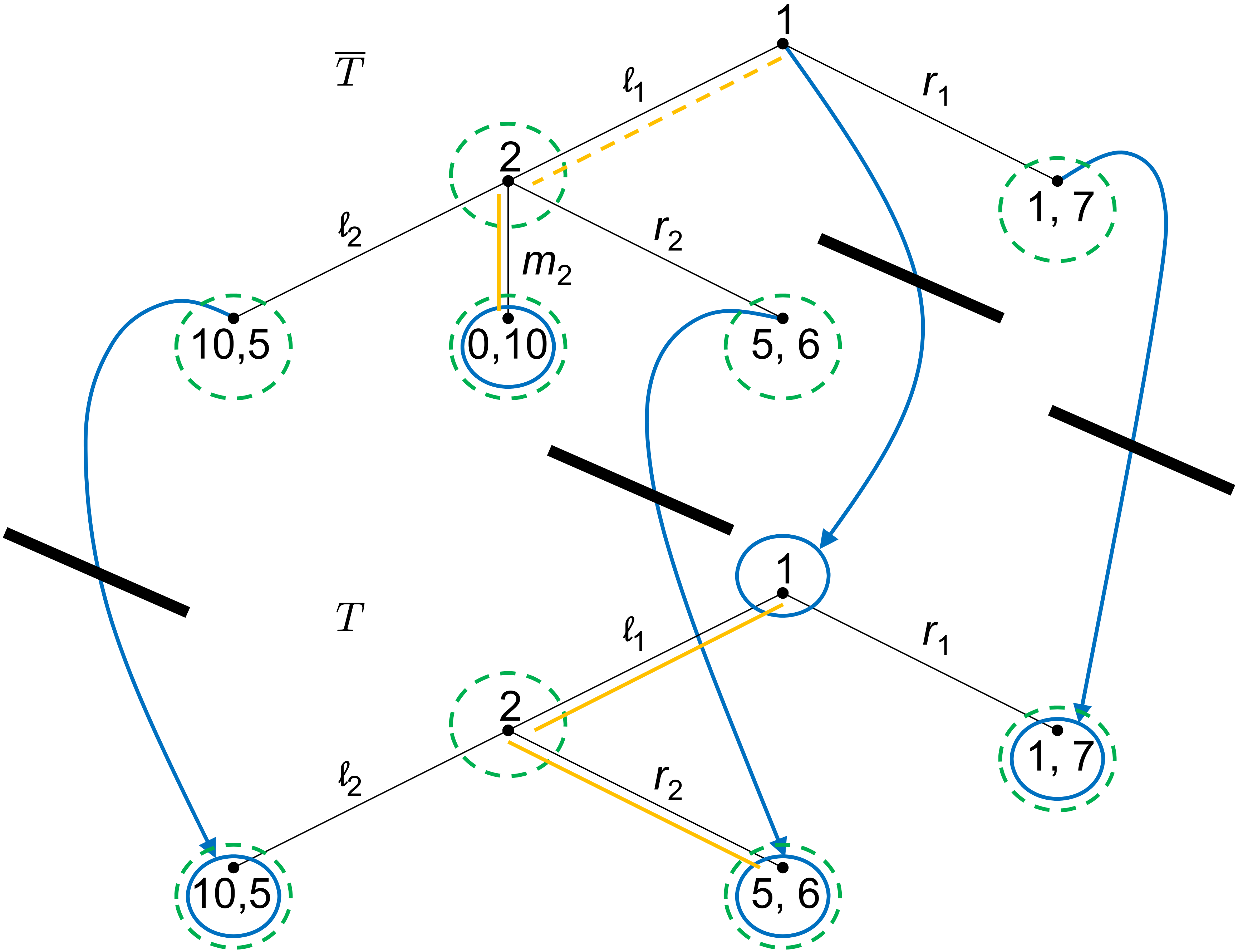
T


$\overline{T}$
1
ℓ_1
r_1
2
1, 7
ℓ_2
m_2
r_2
10,5
0,10
5, 6
T
1
ℓ_1
r_1
2
1, 7
ℓ_2
r_2
10,5
5, 6

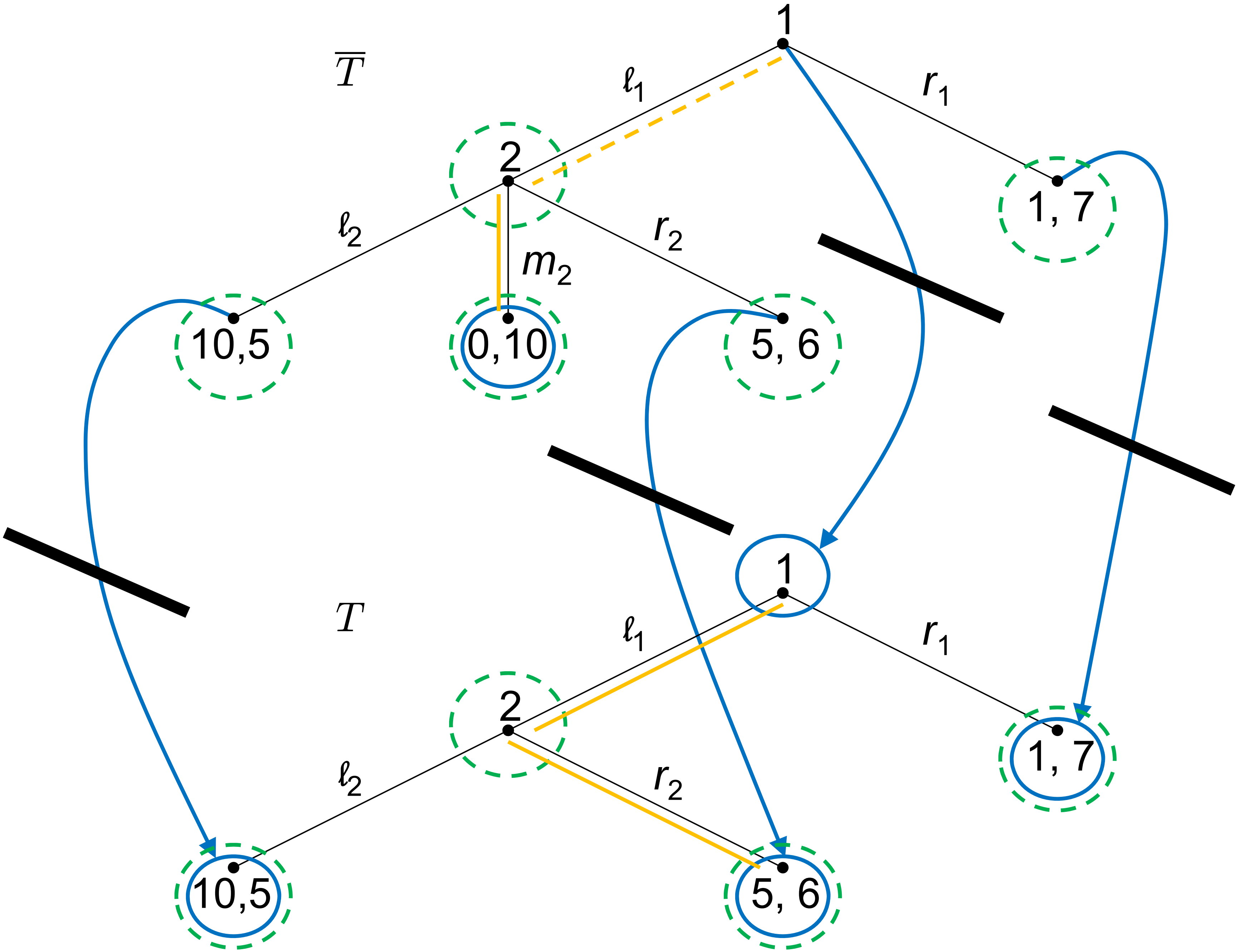

$\overline{T}$
1
ℓ_1
r_1
2
1, 7
ℓ_2
m_2
r_2
10,5
0,10
5, 6
T
1
ℓ_1
r_1
2
1, 7
ℓ_2
r_2
10,5
5, 6

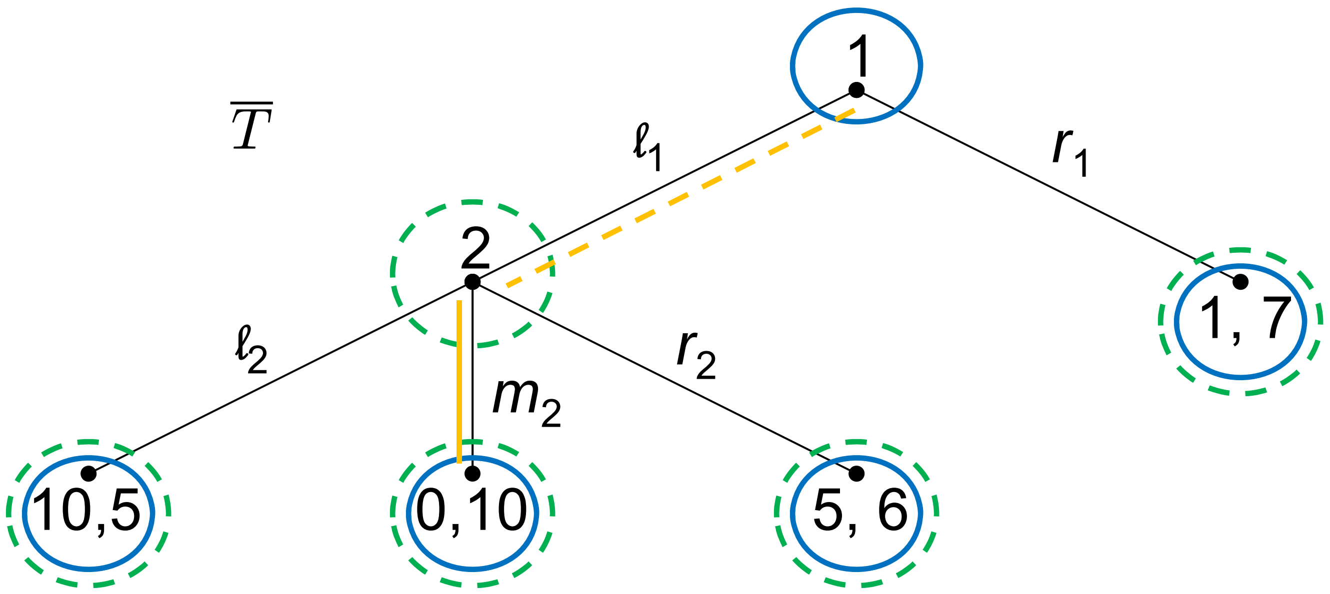

$\overline{T}$
1
ℓ_1
r_1
2
1, 7
ℓ_2
m_2
r_2
10,5
0,10
5, 6
T
1
ℓ_1
r_1
2
1, 7
ℓ_2
r_2
10,5
5, 6

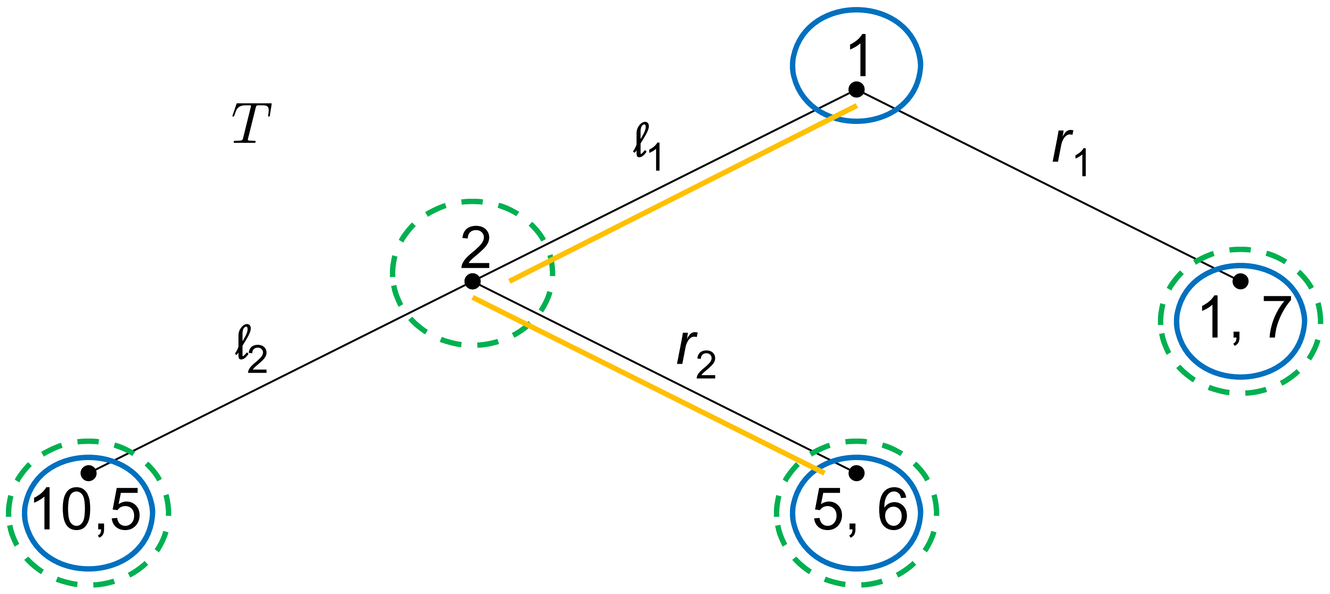

$\overline{T}$
1
ℓ_1
r_1
2
1, 7
ℓ_2
m_2
r_2
10,5
0,10
5, 6
T
1
ℓ_1
r_1
2
1, 7
ℓ_2
r_2
10,5
5, 6

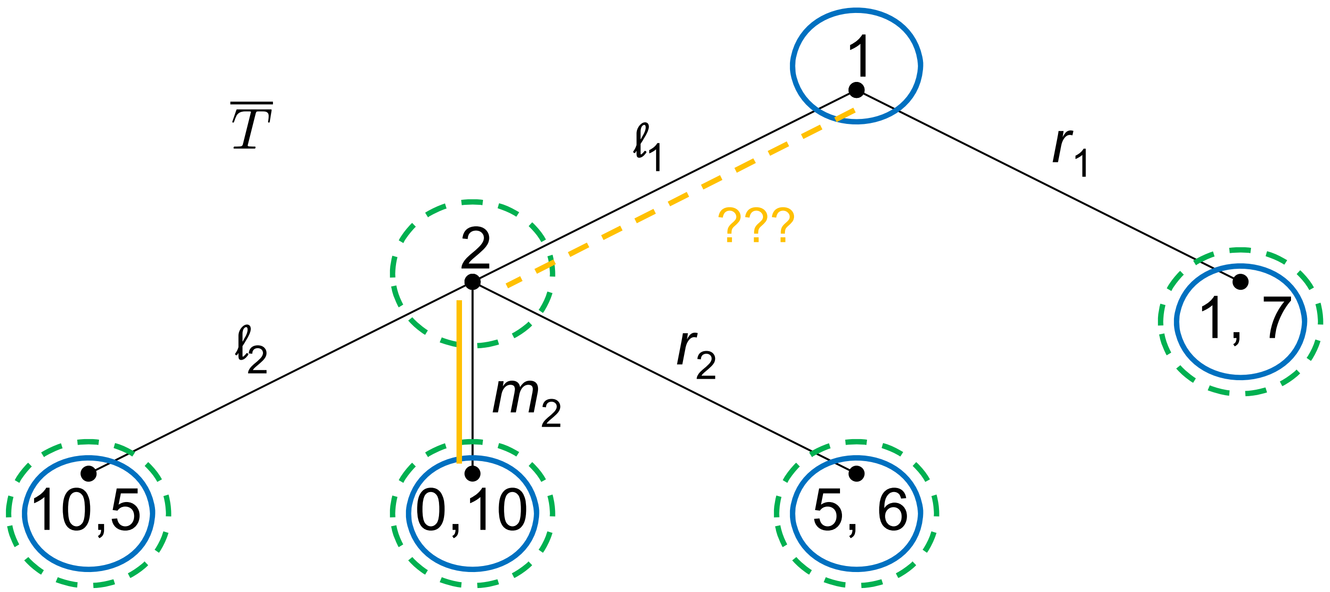


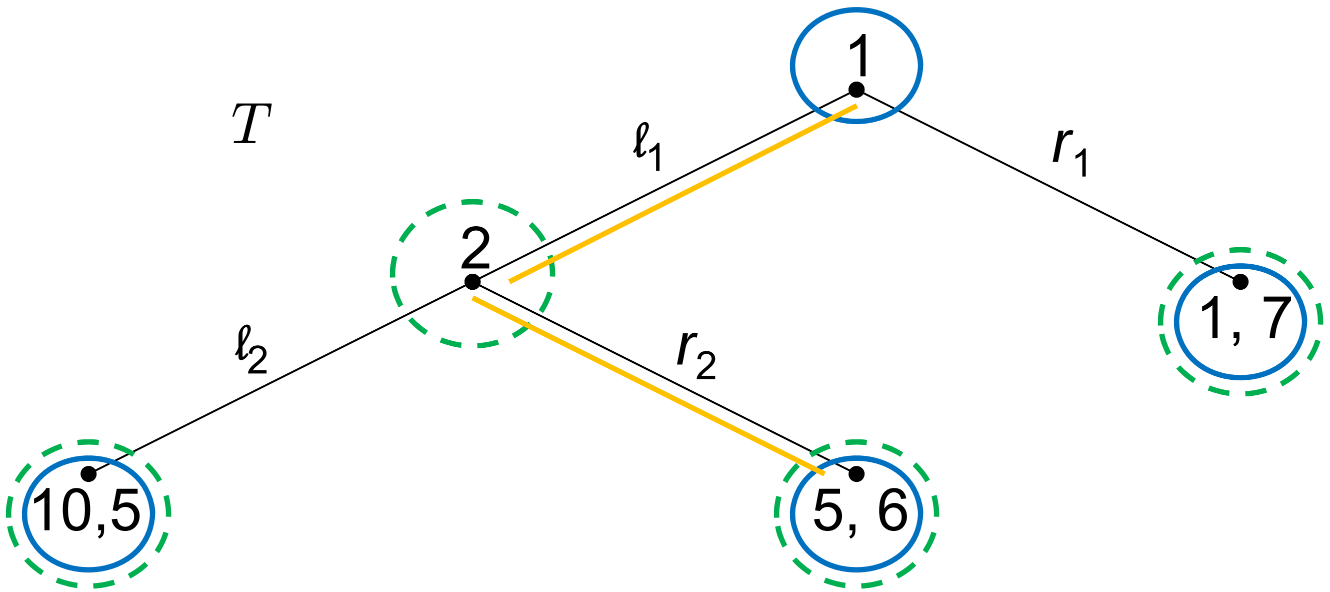
$\overline{T}$
1
ℓ_1
r_1
2
1, 7
ℓ_2
r_2
m_2
10,5
0,10
5, 6
T
1
ℓ_1
r_1
2
1, 7
ℓ_2
r_2
10,5
5, 6

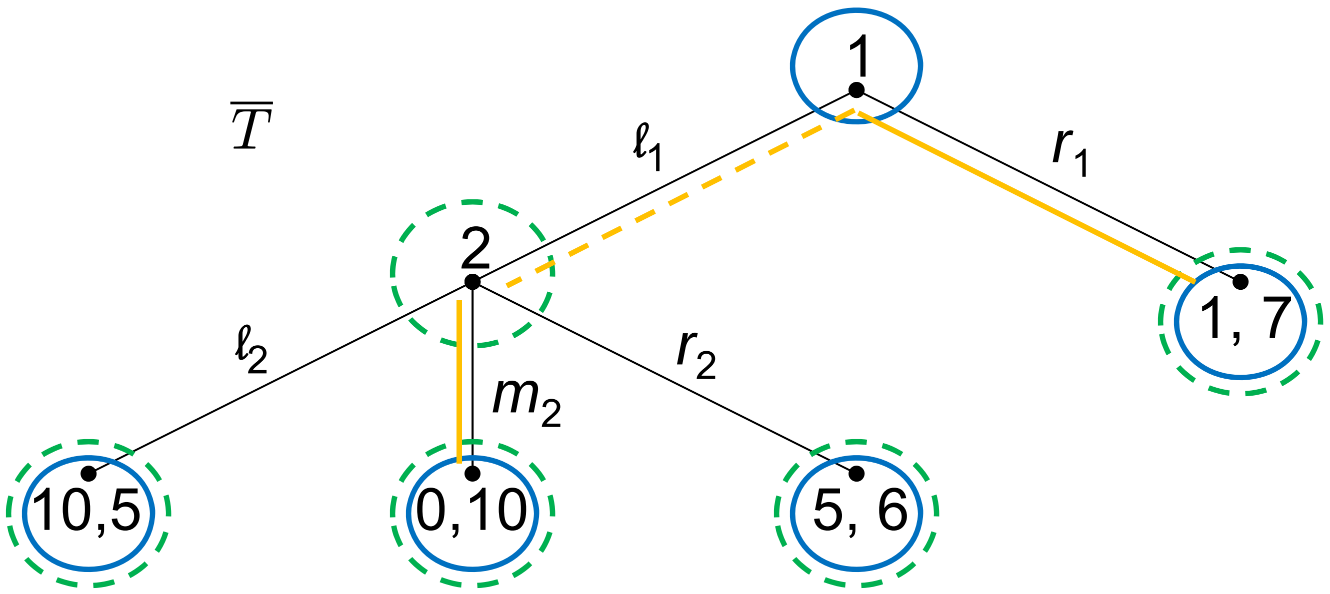

$\overline{T}$
1
ℓ_1
r_1
2
1, 7
ℓ_2
m_2
r_2
10,5
0,10
5, 6
T
1
ℓ_1
r_1
2
1, 7
ℓ_2
r_2
10,5
5, 6

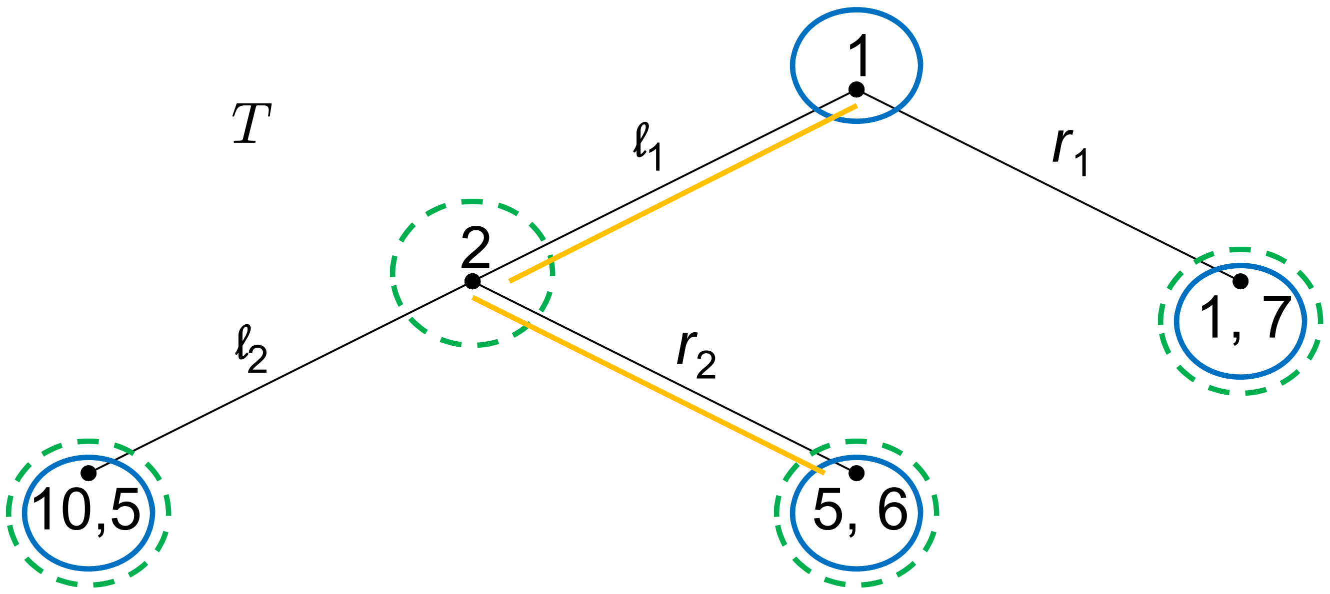

$\overline{T}$
1
ℓ_1
r_1
2
1, 7
ℓ_2
m_2
r_2
10,5
0,10
5, 6


T
1
ℓ_1
r_1
2
1, 7
ℓ_2
r_2
10,5
5, 6

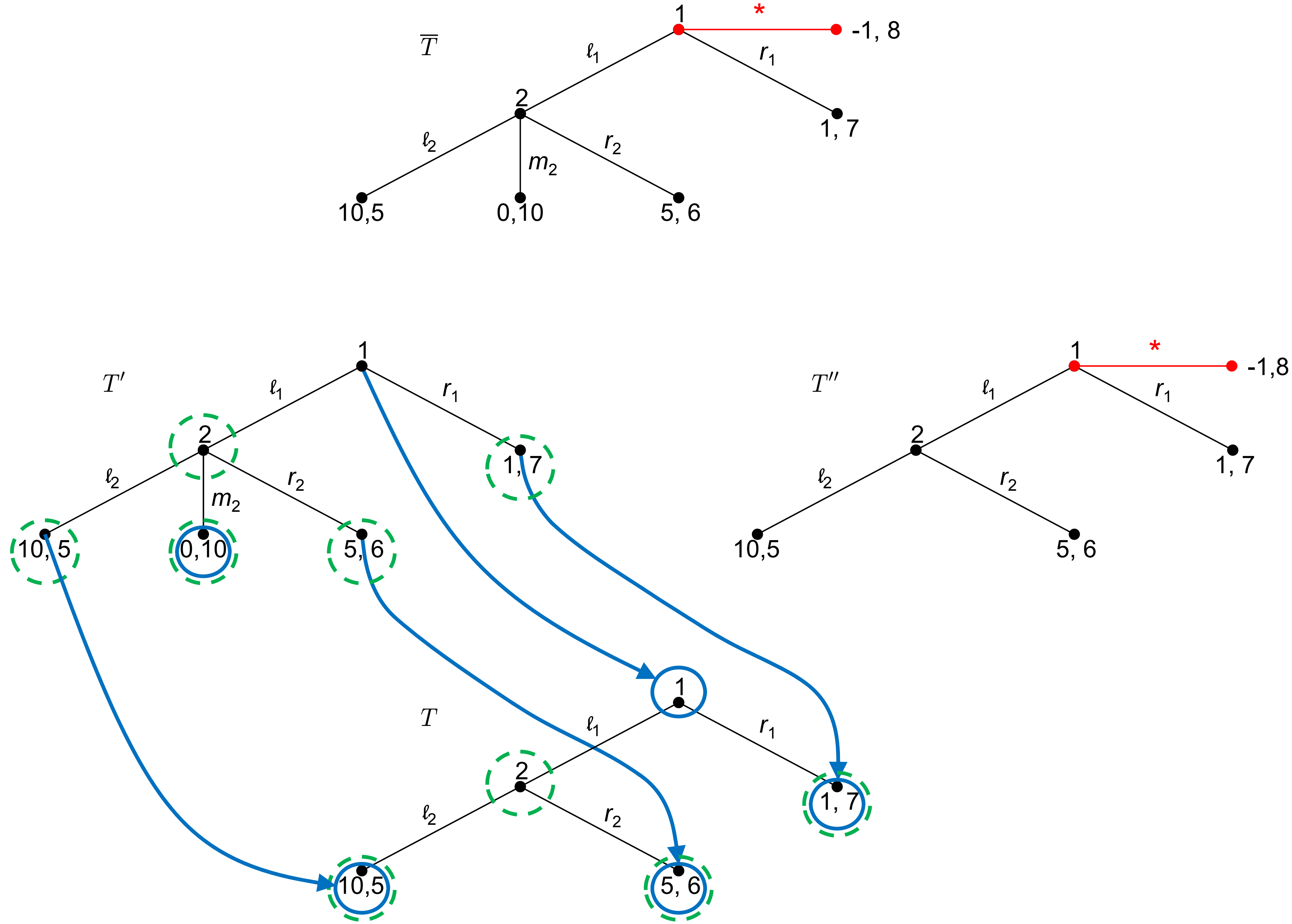

$\overline{T}$
1
ℓ_1
r_1
???
2
ℓ_2
m_2
r_2
1, 7
10,5
0,10
5, 6


T
1
ℓ_1
r_1
2
ℓ_2
r_2
1, 7
10,5
5, 6

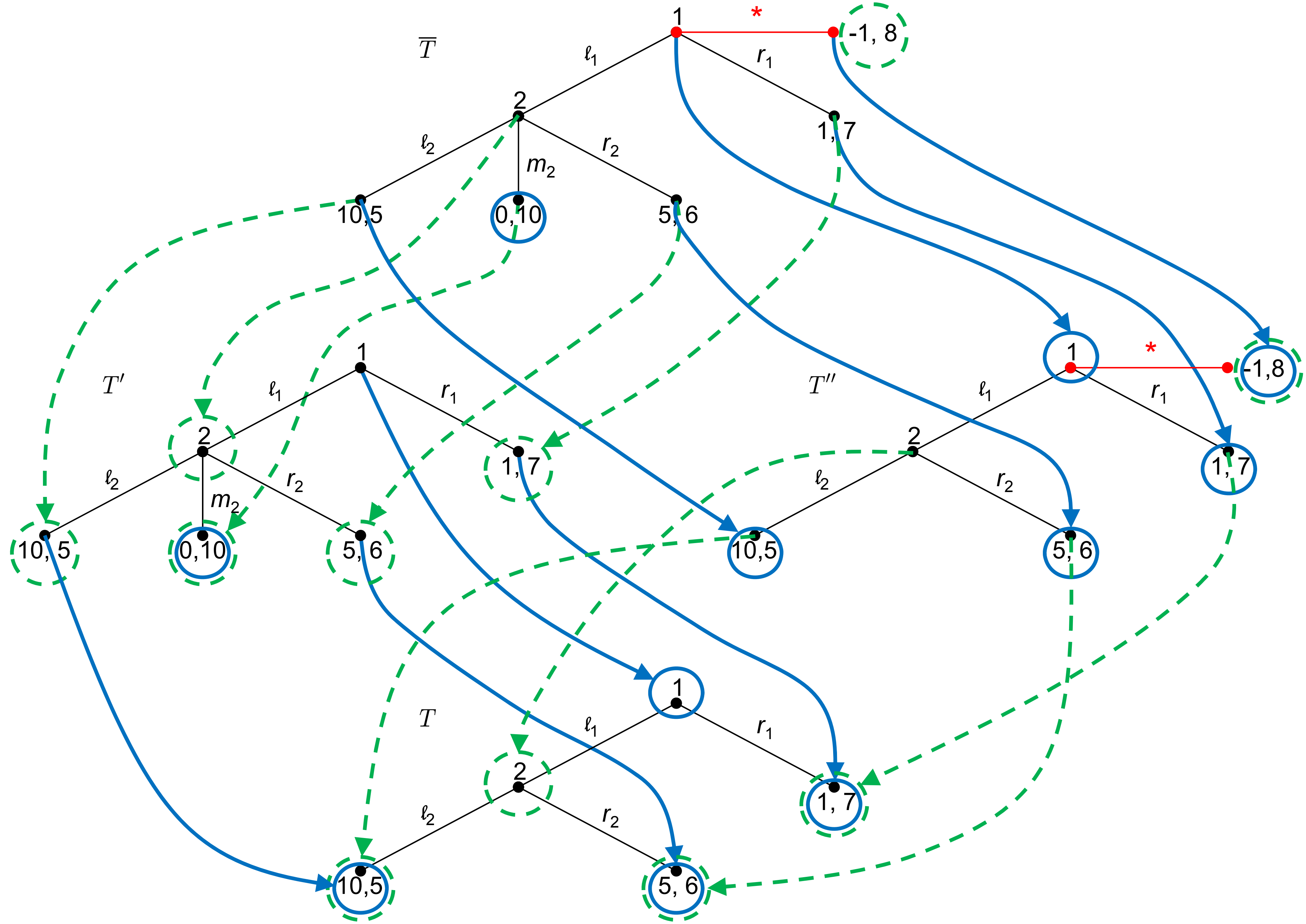

$\overline{T}$
1
ℓ_1
r_1
2
1, 7
ℓ_2
m_2
r_2
10,5
0,10
5, 6


T
1
ℓ_1
r_1
2
1, 7
ℓ_2
r_2
10,5
5, 6

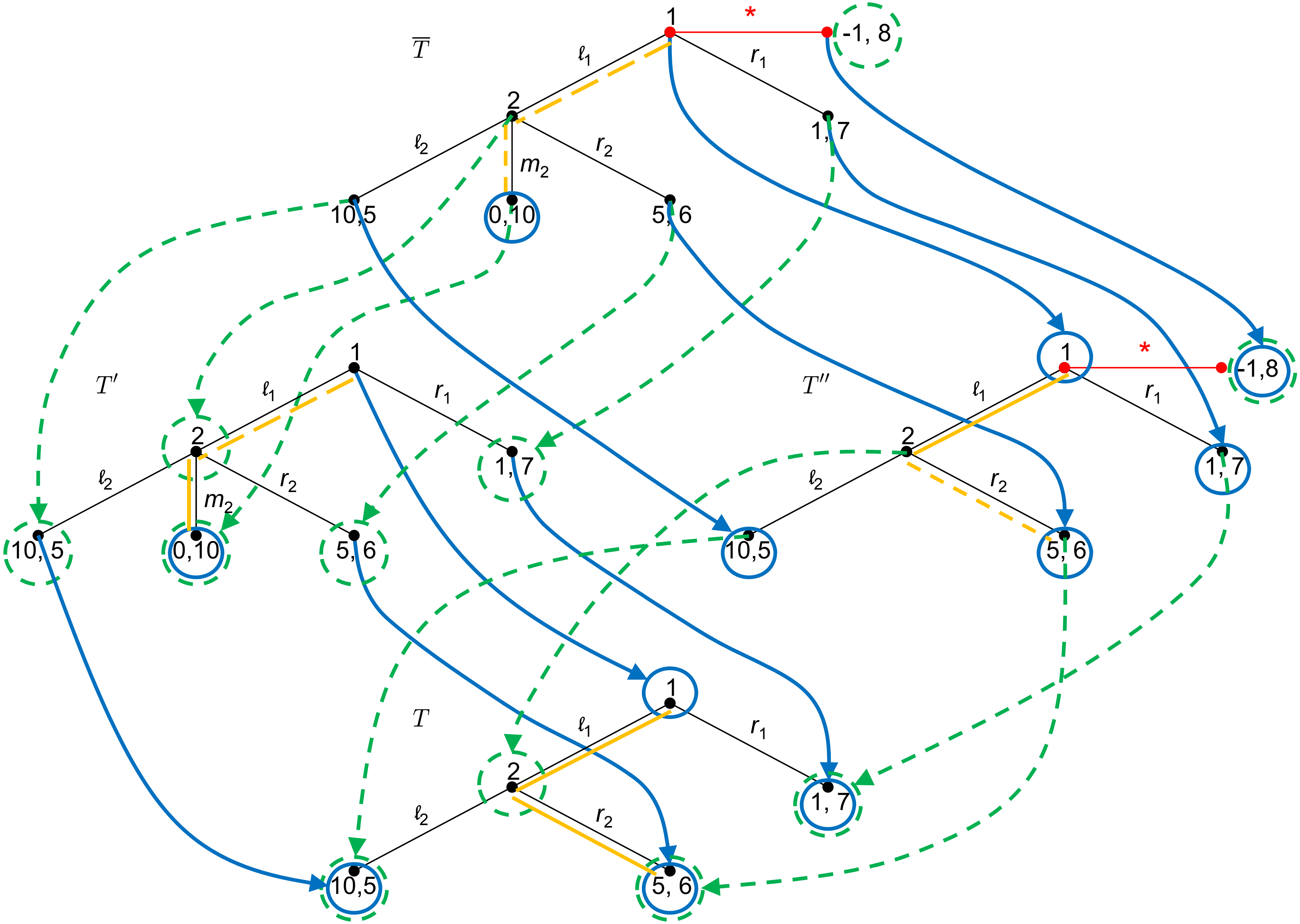
- Does discovery always lead to common awareness?
- Rationalizable discoveries versus non-rationalizable discoveries: Does rational play select among discoveries and thus representations of the strategic context?



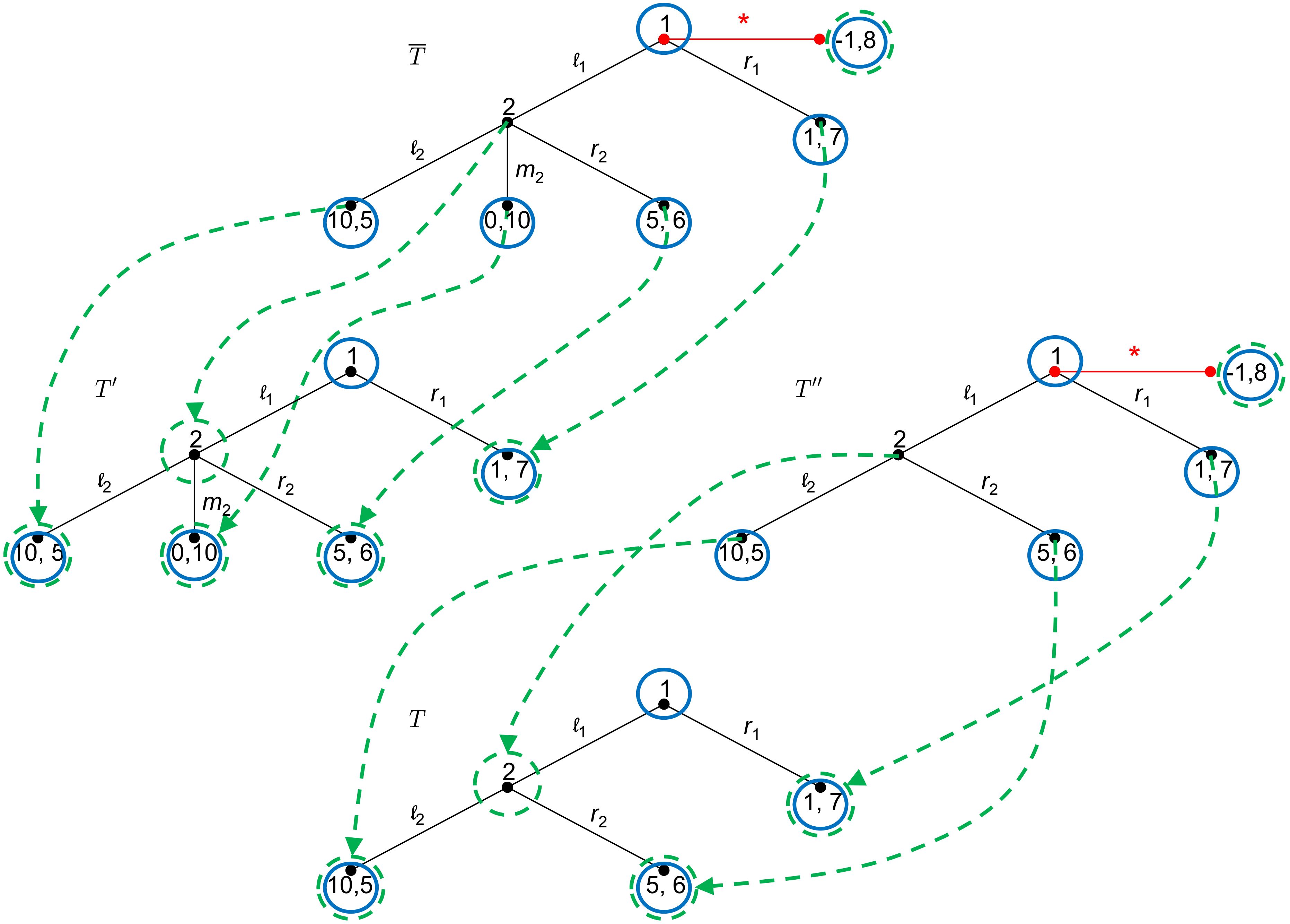
$\overline{T}$
1
*
-1, 8
ℓ_1
r_1
2
1, 7
ℓ_2
m_2
r_2
10,5
0,10
5, 6
T'
1
ℓ_1
r_1
2
1, 7
ℓ_2
m_2
r_2
10, 5
0,10
5, 6
T''
1
*
-1,8
ℓ_1
r_1
2
1, 7
ℓ_2
r_2
10,5
5, 6
T
1
ℓ_1
r_1
2
1, 7
ℓ_2
r_2
10,5
5, 6

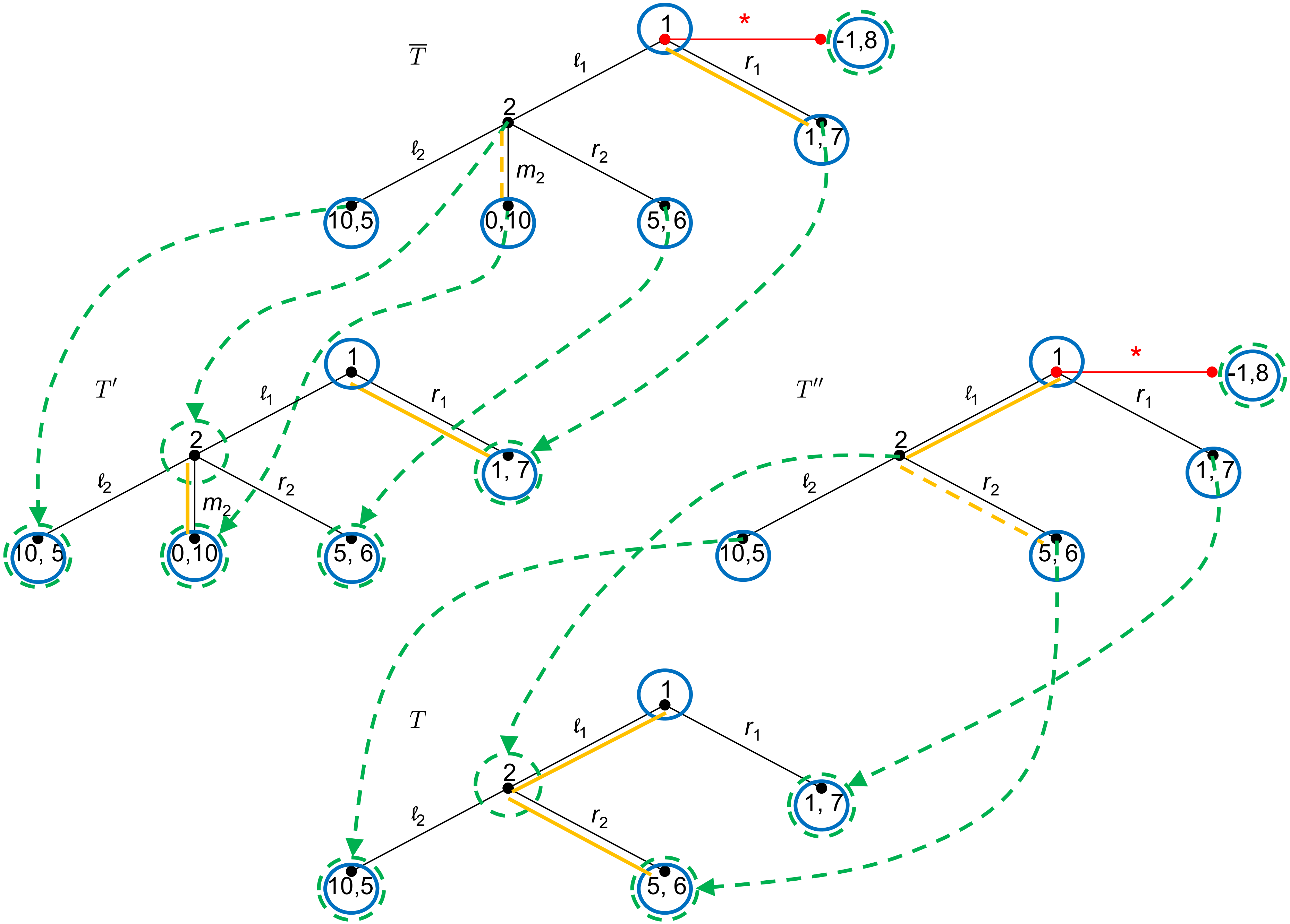


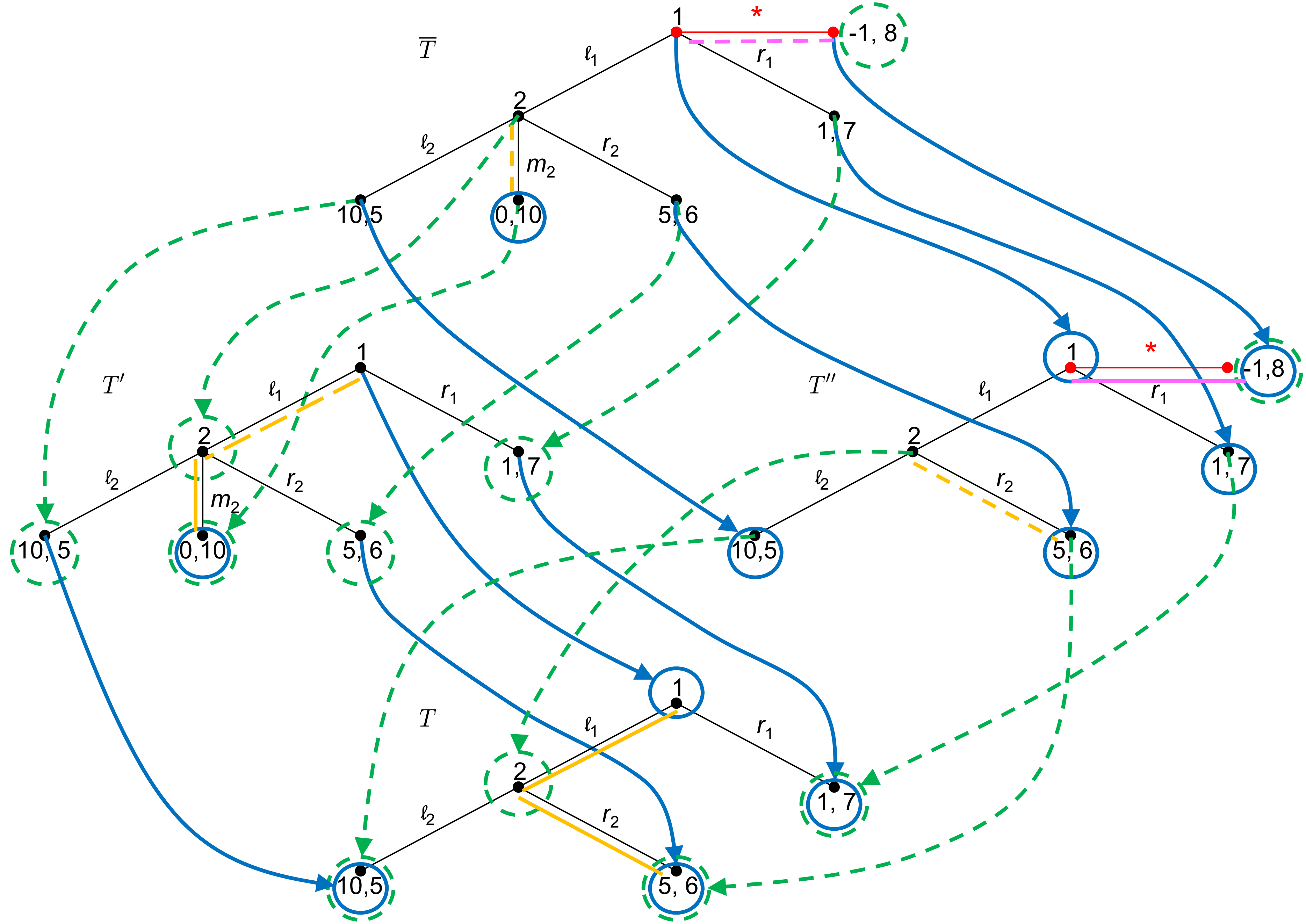
$\overline{T}$
1
*
-1, 8
ℓ_1
r_1
2
1, 7
ℓ_2
m_2
r_2
10,5
0,10
5, 6
T'
1
ℓ_1
r_1
2
1, 7
ℓ_2
m_2
r_2
10, 5
0,10
5, 6
T''
1
*
-1,8
ℓ_1
r_1
2
1, 7
ℓ_2
r_2
10,5
5, 6
T
1
ℓ_1
r_1
2
1, 7
ℓ_2
r_2
10,5
5, 6

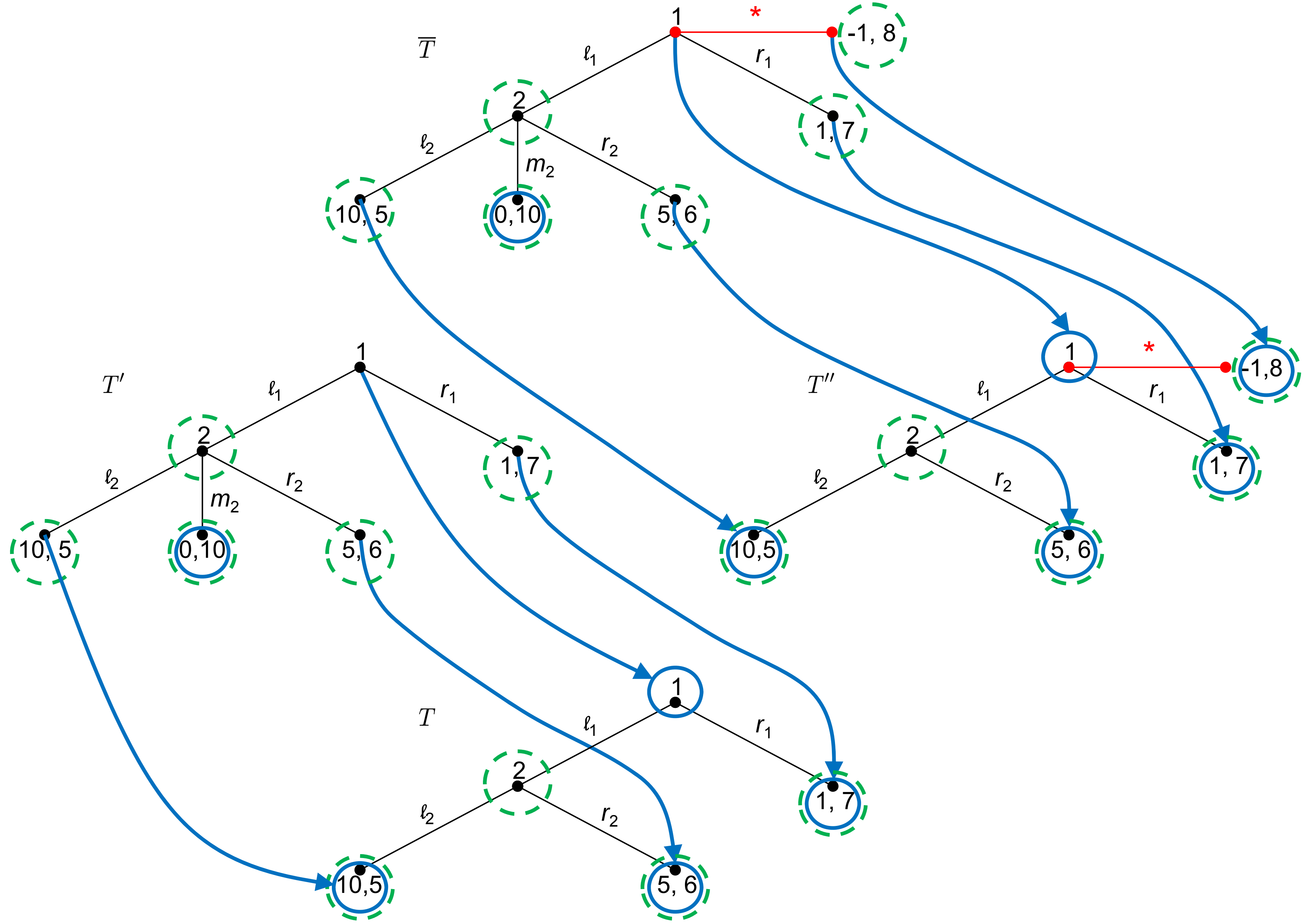


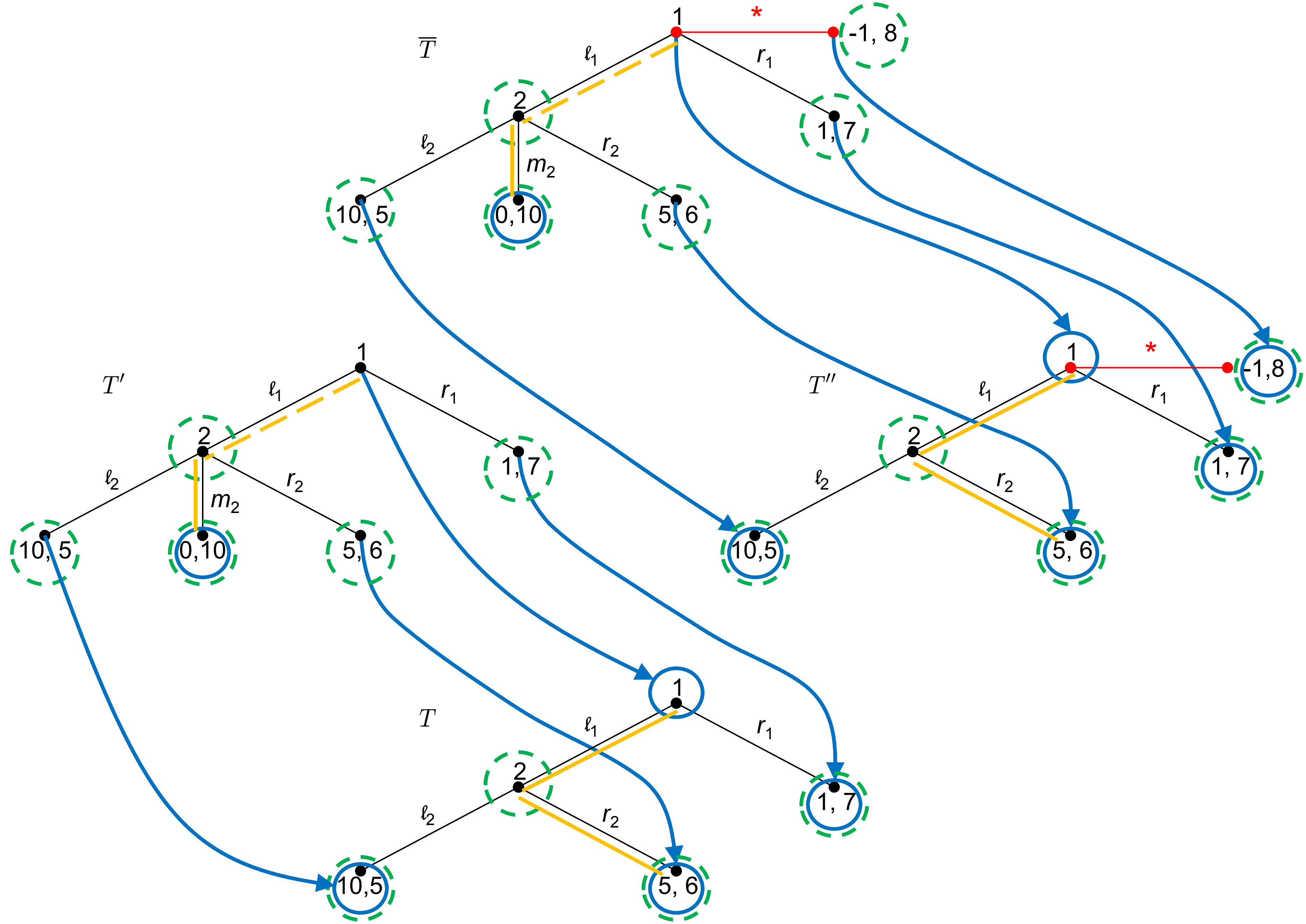
$\overline{T}$
1
*
-1, 8
ℓ_1
r_1
2
1, 7
ℓ_2
m_2
r_2
10,5
0,10
5, 6
T'
1
ℓ_1
r_1
2
1, 7
ℓ_2
m_2
r_2
10, 5
0,10
5, 6
T''
1
*
-1,8
ℓ_1
r_1
2
1, 7
ℓ_2
r_2
10,5
5, 6
T
1
ℓ_1
r_1
2
1, 7
ℓ_2
r_2
10,5
5, 6

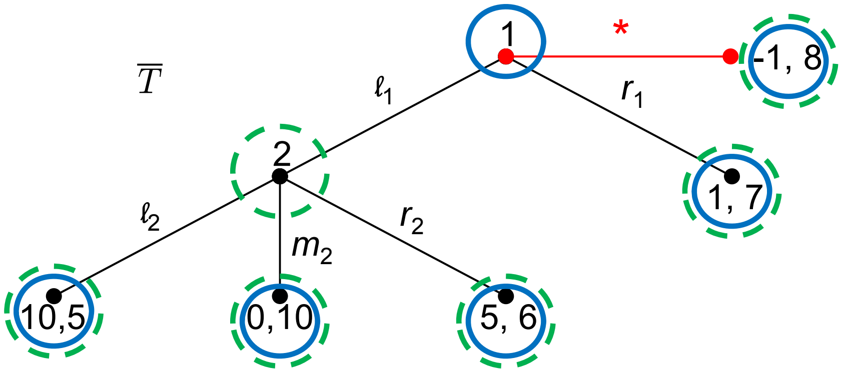

$\overline{T}$
1
*
-1,8
ℓ_1
r_1
2
ℓ_2
m_2
r_2
1, 7
10,5
0,10
5, 6
T'
1
ℓ_1
r_1
2
ℓ_2
m_2
r_2
1, 7
10, 5
0,10
5, 6
T''
1
*
-1,8
ℓ_1
r_1
2
ℓ_2
r_2
1, 7
10,5
5, 6
T
1
ℓ_1
r_1
2
ℓ_2
r_2
1, 7
10,5
5, 6

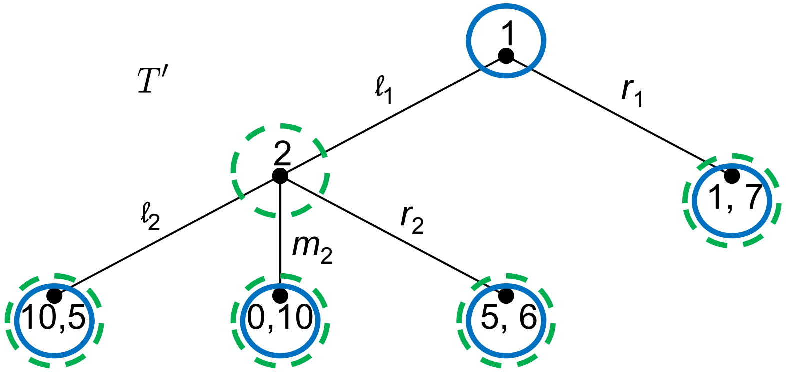

$\overline{T}$
1
*
-1,8
ℓ_1
r_1
2
1, 7
ℓ_2
m_2
r_2
10,5
0,10
5, 6
T'
1
ℓ_1
r_1
2
1, 7
ℓ_2
m_2
r_2
10, 5
0,10
5, 6
T''
1
*
-1,8
ℓ_1
r_1
2
1, 7
ℓ_2
r_2
10,5
5, 6
T
1
ℓ_1
r_1
2
1, 7
ℓ_2
r_2
10,5
5, 6

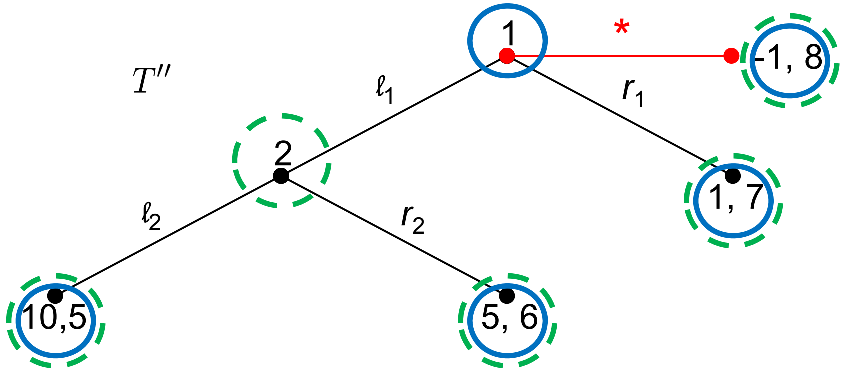

$\overline{T}$
1
*
-1, 8
ℓ_1
r_1
2
1, 7
ℓ_2
m_2
r_2
10,5
0,10
5, 6
T'
1
ℓ_1
r_1
2
1, 7
ℓ_2
m_2
r_2
10, 5
0,10
5, 6
T''
1
*
-1,8
ℓ_1
r_1
2
1, 7
ℓ_2
r_2
10,5
5, 6
T
1
ℓ_1
r_1
2
1, 7
ℓ_2
r_2
10,5
5, 6

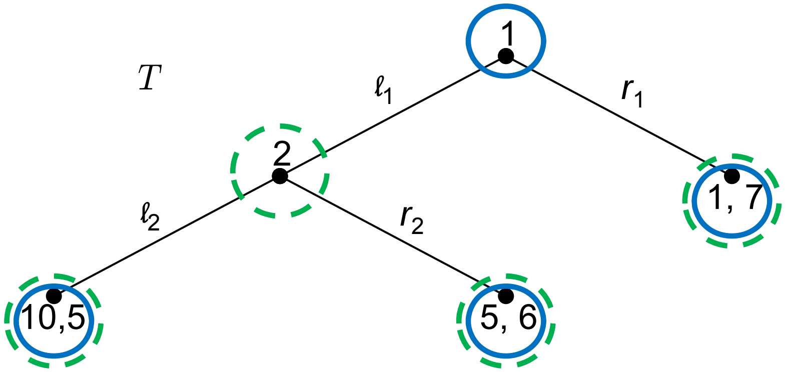

$\overline{T}$
1
*
-1, 8
ℓ_1
r_1
2
1, 7
ℓ_2
m_2
r_2
10, 5
0,10
5, 6
T'
1
ℓ_1
r_1
2
1, 7
ℓ_2
m_2
r_2
10, 5
0,10
5, 6
T''
1
*
-1,8
ℓ_1
r_1
2
1, 7
ℓ_2
r_2
10,5
5, 6
T
1
ℓ_1
r_1
2
1, 7
ℓ_2
r_2
10,5
5, 6

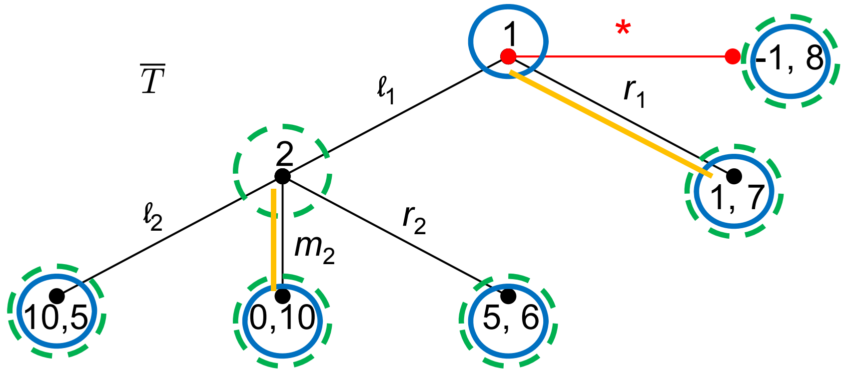

$\overline{T}$
1
*
-1, 8
ℓ_1
r_1
2
1, 7
ℓ_2
m_2
r_2
10, 5
0,10
5, 6
T'
1
ℓ_1
r_1
2
1, 7
ℓ_2
m_2
r_2
10, 5
0,10
5, 6
T''
1
*
-1,8
ℓ_1
r_1
2
ℓ_2
r_2
1, 7
10,5
5, 6
T
1
ℓ_1
r_1
2
ℓ_2
r_2
1, 7
10,5
5, 6


$\overline{T}$
1
*
-1, 8
ℓ_1
r_1
2
1, 7
ℓ_2
m_2
r_2
10,5
0,10
5, 6

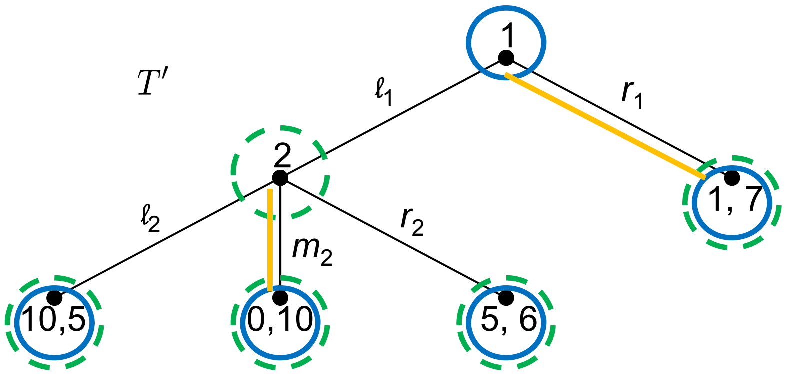

T'
1
ℓ_1
r_1
2
1, 7
ℓ_2
m_2
r_2
10,5
0,10
5, 6


T''
1
*
-1, 8
ℓ_1
r_1
2
1, 7
ℓ_2
r_2
10,5
5, 6

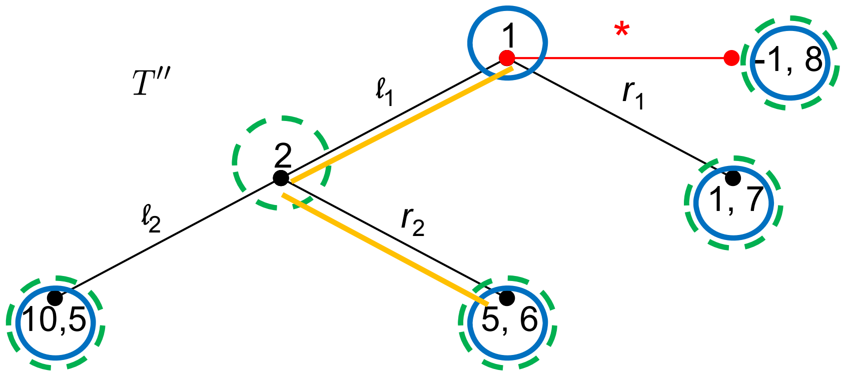

T
1
ℓ_1
r_1
2
1, 7
ℓ_2
r_2
10,5
5, 6



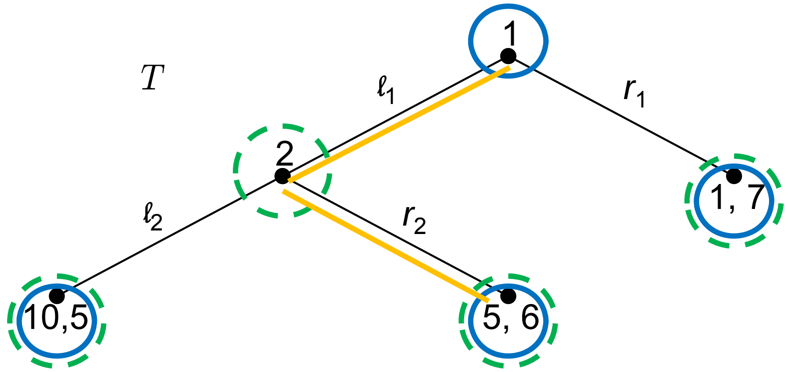
$\overline{T}$
1
*
-1, 8
ℓ_1
r_1
2
1, 7
ℓ_2
m_2
r_2
10,5
0,10
5, 6



T'
1
ℓ_1
r_1
2
1, 7
ℓ_2
m_2
r_2
10,5
0,10
5, 6



T''
1
*
-1, 8
ℓ_1
r_1
2
1, 7
ℓ_2
r_2
10,5
5, 6



T
1
ℓ_1
r_1
2
1, 7
ℓ_2
r_2
10,5
5, 6

- Consider extensive-form games with unawarenes a la Heifetz, Meier, Schipper (2013) with the stronger properties of unawareness.
- Require additionally that the set of trees is a lattice.
- Devise information sets also at terminal nodes.

Self-confirming equilibrium

Profile of behavioral strategies satisfying: For each player

(0) awareness is self-confirming (i.e. constant) along the path(s)

(i) is rational along the path(s)

(ii) beliefs are self-confirming (constant and correct) along the path(s)

Mutual knowledge of no changes of awareness?

Lemma: No change of awareness for each player implies "mutual knowledge" of no change of awareness.

Proof: Follows from complete lattice of trees and conditions on info. sets.

Existence of self-confirming equilibrium

- exists in standard games
- often fails in games with unawareness (because of changes of awareness implied by rational play)

A game is a *self-confirming game* if it possesses a self-confirming equilibrium.

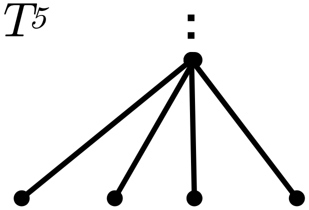
Remark: Every game with common constant awareness is a self-confirming game. Converse is false.

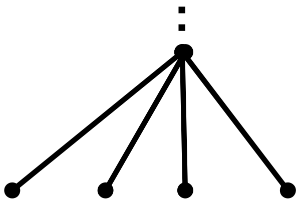
Discovered version

Given an extensive-form game with unawareness and a strategy profile, the discovered version is defined by

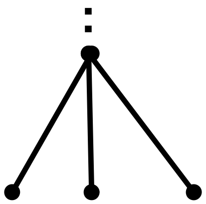
- the same set of players, same lattice of trees, the same player function, same payoffs
- “updated” information sets:

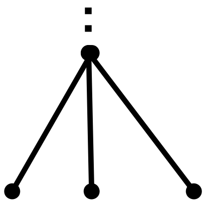
T^5



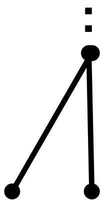


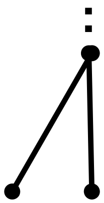
T^4



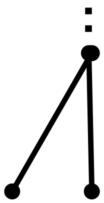


T^3

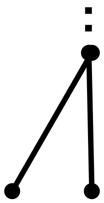


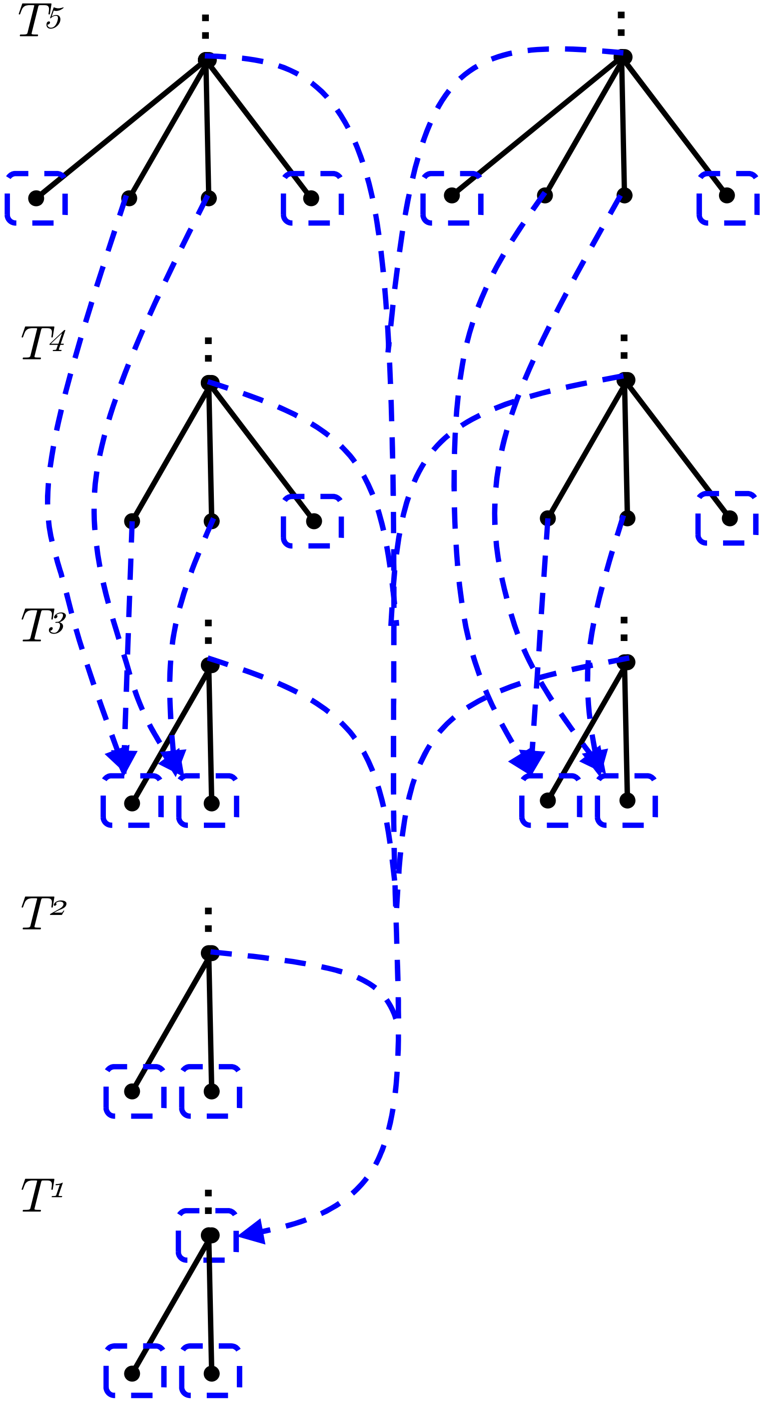


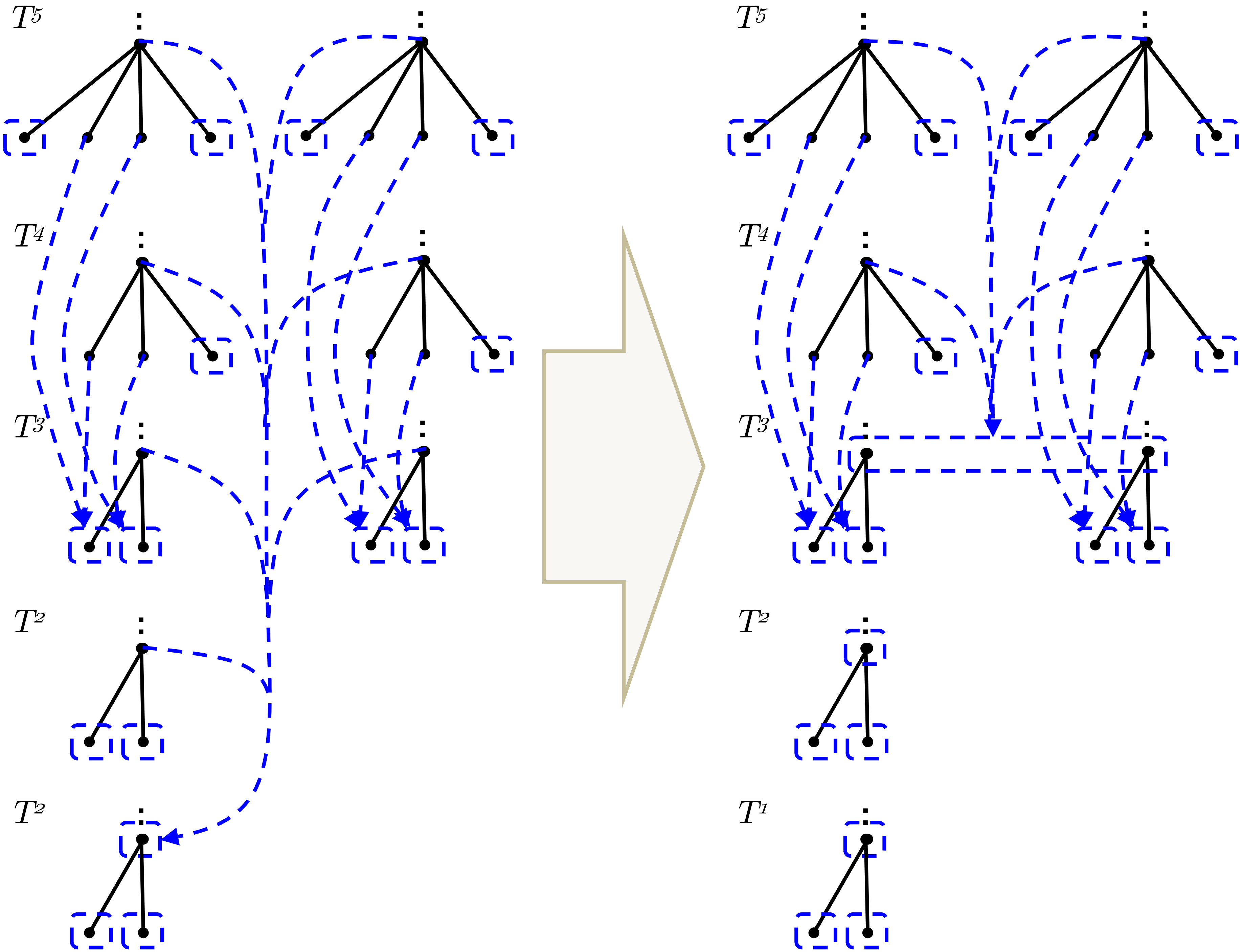
T^2



T^1




T^5
T^4
T^3
T^2
T^1



T^5
T^4
T^3
T^2
T^2
T^5
T^4
T^3
T^2
T^1

Discovered version

Discovered versions are well-defined:

Lemma: For any extensive-form game with unawareness and strategy profile, the discovered version is an extensive-form game with unawareness.

Lemma: Given an extensive-form game with unawareness, any two strategy profiles that generate the same path have identical discovered versions.

Discovery process

Finite state machine:

- Set of states: set of extensive-form games with unawareness (with identical upmost tree)
- Initial state/game
- Output function assigns strategy profile to each game
- Transition function assigns the discovered version to each game and strategy profile
- Final states/games (absorbing)

Discovery process

Proposition: For every discovery process, the final state exists and is unique.

Remark: Final states may not have common constant awareness.

Proposition: For every extensive-form game with unawareness, there exists a discovery process that leads to a self-confirming game.

Remark: Typically extensive-form games with unawareness have several self-confirming versions.

Extensive-form rationalizability

Inductive definition

Level 0

- All belief systems
- Any strategy for which there exists a belief system such that the strategy is optimal at every non-terminal info. set.

Level k

- All belief systems concentrated on k-1 level rationalizable strategies of opponents if possible.
- Any strategy for which there exists a level-k belief system such that the strategy is optimal at every non-terminal info. set.

Extensive-form rationalizable strategies: intersection of every level-k rationalizable strategy sets.

Rationalizable discoveries

A discovery process is rationalizable if the output function assigns to each game an extensive-form rationalizable strategy.

Proposition: For every extensive-form game with unawareness, there exists a rationalizable discovery process whose final state is a self-confirming version. This refines the set of self-confirming versions.

Rationalizable self-confirming equilibrium

A self-confirming equilibrium is rationalizable if non-rationalizable paths of play have zero probability. (???)

Conjecture: For every extensive-form game with unawareness, there exists a rationalizable discovery process whose final state is a game with a rationalizable self-confirming equilibrium.

Applications of unawareness

- **Dynamic games:** Halpern and Rego (MSS 2014), Rego and Halpern (IJGT 2012), Heifetz, Meier, and Schipper (GEB 2013), Feinberg (2004, 2005, 2012), Grant and Quiggin (ET 2013), Li (2006), Ozbay (2008), Nielsen and Sebald (2013)
- **Bayesian games with unawareness:** Meier and Schipper (ET 2014), Sadzik (2006)
- **Value of information, value of awareness:** Galanis (2013), Quiggin (2013)
- **Disclosure of information:** Heifetz, Meier, and Schipper (2012), Li, Peitz, and Zhao (MSS 2014), Schipper and Woo (2015)
- **Contract theory:** Auster (GEB 2013), Filiz-Ozbay (GEB 2012), Grant, Kline, and Quiggin (JEBO 2012), Lee (2008), Zhao and van Thadden (RES 2013)
- **Decision theory:** Karni and Viero (AER 2013), Schipper (IJGT 2013), Li (2006)
- **Electoral Campaigning:** Schipper and Woo (2015)
- **General equilibrium:** Exchange economies: Modica, Rustichini, and Thallon (ET 1998), Production economies: Kawamura (JET 2005)
- **Finance:** Siddiqi (2014)
- … (need more)

I am looking forward to receiving your paper on unawareness.