Quantum Logic

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Hilbert Spaces and Quantum Theory

Hilbert spaces have been used for modeling quantum systems. This is largely because wave functions live in Hilbert spaces, and Hilbert spaces give rise to probabilities that match our understanding of quantum phenomena.

Hilbert Space and Inner Product Space

Here are some terse definitions (to be explained further in coming slide):

Definition (Hilbert space)

A Hilbert space is a complete inner product space.

What is an inner product space?

Definition

An inner product space is a vector space endowed with an inner product.

Inner product

Definition (Inner product)

Given a vector space V over the complex numbers $\mathbb{C}$, an inner product is a function $\langle \cdot, \cdot \rangle : V \to \mathbb{C}$, such that

1. $\langle v, cw + x \rangle = c\langle v, w \rangle + \langle v, x \rangle$
   [$\langle \cdot, \cdot \rangle$ is linear in its second coordinate]
2. $\langle w, v \rangle = \overline{\langle v, w \rangle}$ [Conjugate symmetry]
   (where for any complex number $a + bi$ (with $a, b \in \mathbb{R}$), $\overline{a + bi} = a - bi$ is its complex conjugate)
3. $\langle v, v \rangle \geq 0$ [Non-negativity]
4. $\langle v, v \rangle = 0$ if and only if $v = 0$. [Positive-definiteness]

Interpretation of an inner product

An inner product measures much of the geometric structure of a Hilbert space:

- "angles" between vectors:
  v is orthogonal to w iff both $v \neq w$ and $\langle v, w \rangle = 0$.
- "length" of vectors
  Induced norm: $\|v\| = \sqrt{\langle v, v \rangle}$
- "distance" between vectors
  Induced metric: $\mu(v, w) = \|v - w\|$.

The inner product also gives rise to probabilities that a quantum state with vector v collapses to a quantum state with vector w when asked about w:

$$Pr_w(v) = \frac{|\langle v, w \rangle|^2}{\|v\|^2 \|w\|^2}$$

What is a norm?

Definition (Norm)

Given a vector space V of a field $F \subseteq \mathbb{C}$, a norm is a function $\|\cdot\| : V \to \mathbb{R}$, such that

1. $\|c \cdot v\| = |c| \cdot \|v\|$ for $c \in F$ and $v \in V$.
2. $\|v + w\| \leq \|v\| + \|w\|$ [triangle inequality]
3. $\|v\| = 0$ implies $v = 0$

What is a metric?

Definition (Metric)

Given any set X, a metric on X is a function $\mu : X \times X \to \mathbb{R}$, such that

1. $\mu(x, y) = \mu(y, x)$ [symmetry]
2. $\mu(x, y) \leq \mu(x, z) + \mu(y, z)$ [triangle inequality]
3. $\mu(x, y) \geq 0$ [non-negativity]
4. $\mu(x, y) = 0$ if and only if $x = y$ [positive definiteness]

What is completeness of an inner-product space?

Recall an inner product induces a metric.

Definition (Cauchy sequence)

Given a metric $\mu : X \to \mathbb{R}$, a Cauchy sequence is an X-valued sequence (a_n), such that for every $\epsilon > 0$, there is an N, such that for all $n, m \geq N$, $\mu(a_n, a_m) < \epsilon$.

Definition (Convergence)

An X-valued sequence (a_n) converges to a if for every $\epsilon > 0$, there exists N, such that for all $n \geq N$, $\mu(a_n, a) < \epsilon$. A sequence converges if it converges to a for some a.

Every convergent sequence is Cauchy, but not every Cauchy sequence converges.

Definition (Complete)

A metric is complete if every Cauchy sequence converges.
An inner product space is complete if its metric is complete.

Recap and closed linear subspace

- An inner product space induces a metric.
- The inner product space is Hilbert space if every Cauchy sequence converges.

We are particularly interested in topologically closed linear subspaces of a Hilbert space.

But what does a topology have to do with a Hilbert space?

What is a topology?

Definition (Topology)

A topological space is a pair (X, τ), where X is a set and $\tau \subseteq \mathcal{P}(X)$, such that

1. $X, \emptyset \in \tau$
2. Arbitrary unions of elements of τ are in τ
3. Finite intersections of elements of τ are in τ

τ consists of open sets and the complement of an open set is a closed set.

Definition (Topological Closure)

For any subset of S of X, let $\mathrm{cl}(S)$ be the closure of S, smallest closed set containing S.

What is a metric space?

Definition (Metric space)

A metric space is a pair (X, μ), where X is a set and $\mu : X \times X \to \mathbb{R}$ is a metric.

A metric μ induces the smallest topology on X containing

$$\{\{y \mid \mu(x, y) < r\} \mid x \in X, r \in \mathbb{R}\}.$$

Example

$X = \mathbb{R}$ and $\mu(x, y) = |x - y|$. The open sets are generated by open intervals:

$$\{(a, b) \mid a, b \in \mathbb{R}\}$$

A typical closed set is a closed interval:

$$[a, b]$$

Recap on Hilbert space and topology

- An inner product space induces a metric.
- The metric induces a topology.
- We are concerned with closed linear subspaces of Hilbert spaces.

What is special about closed linear subspaces?

Closed linear subspaces can be identified with projectors, which act as quantum tests.

Closed linear subspaces are testable properties.

What is a projector?

An adjoint of a linear map is A is a linear map $A^\dagger$, such that for any vectors v, w

$$\langle v, Aw \rangle = \langle A^\dagger v, w \rangle$$

If A is represented by a matrix with complex entries, then the matrix representation of $A^\dagger$ is the conjugate transpose of A.

Definition (Projector)

A projector is a linear map A, such that

1. $A = A^\dagger$ [A is Hermitian]
2. $A = A \circ A$ [A is idempotent]
3. $\|A(v)\| \leq c\|v\|$ [A is bounded]

A projector onto P effectively strips away the components of the input not in P and fixes what is in P.

Closed linear subspaces and orthogonality

Notation for orthogonality

For $s, t \in \mathcal{H}$, write $s \perp t$ for $\langle v, w \rangle = 0$.

For $s \in \mathcal{H}$ and $T \subseteq \mathcal{H}$,

$$s \perp T \text{ iff } s \perp t \text{ for every } t \in T.$$

Orthogonality as as unary operator

For any set $S = \mathcal{H}$, let

$$S^{\perp} = \{t \mid t \perp S\}.$$

Projection theorem

Theorem (Projection theorem)

Given a closed linear subspace S of a Hilbert space $\mathcal{H}$, for each $x \in \mathcal{H}$, there is a closest point y to x (using the induced metric) such that $y \in S$. Furthermore, y is the unique element of S that has the property that $(x - y) \perp S$

The following hold.

Proposition

1. *For any set $S \subseteq \mathcal{H}$, $S^\perp$ is a closed linear subspace.*
2. *For any closed linear subspace S, $S = (S^\perp)^\perp$.*

Basis and Dimension

Given a set S of vectors, the finite linear span of S is

$$\mathrm{sp}(S) = \{a_1 x_1 + \cdots + a_n x_n \mid n \in \mathbb{N}, x_i \in S, a_i \in \mathbb{C}\}$$

Definition (Orthonormal basis)

In a Hilbert space, an *orthonormal basis* is a set $\mathcal{B}$, such that

1. $a \perp b$ for all $a, b \in \mathcal{B}$ [$\mathcal{B}$ is orthogonal]
2. $\|a\| = 1$ for all $a \in \mathcal{B}$ [$\mathcal{B}$ consists of unit vectors]
3. $\mathcal{H} = \mathrm{cl}(\mathrm{sp}(\mathcal{B}))$ ($\mathrm{cl}(\mathrm{sp}(\mathcal{B})) = \mathrm{sp}(\mathcal{B})$ if $\mathcal{B}$ is finite)

Every basis of a Hilbert space has the same cardinality.

Definition (Dimension)

The dimension of a Hilbert space is the cardinality of one of its bases.

Properties of finite dimensional Hilbert spaces

Proposition

Every finite dimensional inner-product space is a Hilbert space, and all n-dimensional Hilbert spaces over a field F are isomorphic.

Proposition

Every linear subspace of a finite dimensional Hilbert space is closed.

Hilbert lattice

Given a Hilbert space $\mathcal{H}$, the structure $L(\mathcal{H}) = (X, \subseteq, (\cdot)^\perp)$ with X the closed linear subspaces of $\mathcal{H}$ is a lattice with involution. For any two subspaces $A, B \in L(\mathcal{H})$,

1. The greatest lower bound is
$$A \wedge B = A \cap B$$
2. The least upper bound is
$$A \vee B = \mathrm{cl}(A + B) = \mathrm{cl}(\{a + b \mid a \in A, b \in B\}).$$
3. Also,
$$A \vee B = (A^\perp \wedge B^\perp)^\perp$$

If $\mathcal{H}$ is finite dimensional, $L(\mathcal{H})$ has similar structure to:

- the Grassmanian (a structure consisting of the k-dimensional subspaces of a Hilbert space) for each k
- projective geometries (the “points” of the projective geometry are the “lines” or one-dimensional subspaces of a vector space)

Literature on logics over Hilbert Lattices

- G. Birkhoff and J. von Neumann. The Logic of Quantum Mechanics. *Annals of Mathematics* **37**, pp. 823–843, 1936.
  First paper on quantum logic
- J.M. Dunn, T. Hagge, L.S. Moss, Z. Wang. Quantum Logic as Motivated by Quantum Computation. *The Journal of Symbolic Logic* **70**(2), pp. 353–359, 2005.
  Decidability of first order theory of finite dimensional Hilbert lattices.
- C. Herrmann and M. Ziegler. Computational Complexity of Quantum Satisfiability. In the proceedings of the 26th Anual IEEE Symposium on Logic in Computer Science (LICS), pp.175–184, 2011.
  Complexity of propositional Hilbert quantum logics

Basic quantum logic language

Many quantum logics use the same language as classical propositional logic:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi$$

where $p \in \mathrm{AtProp}$ is a set of atomic proposition letters.
Some abbreviations:

- $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$
- $\varphi \to \psi := \neg\varphi \vee \psi$
- $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$

Propositional Hilbert Quantum Logic semantics

A Hilbert realization of quantum logic is a pair $(\mathcal{H}, V)$, where

- $\mathcal{H}$ is a Hilbert space over set of vectors H, and
- $V : \mathrm{AtProp} \to L(\mathcal{H})$ is a valuation function

We extend V to all propositional formulas as follows:

- $V(\neg\varphi) = V(\varphi)^{\perp}$
- $V(\varphi \wedge \psi) = V(\varphi) \cap V(\psi)$

φ is weakly true in a realization $(\mathcal{H}, V)$ if $V(\varphi) \neq \emptyset$.
φ is strongly true if $V(\varphi) = H$.
φ is weakly (strongly) satisfiable in a Hilbert space $\mathcal{H}$, if there is a valuation V, such that φ is weakly (strongly) true in $(\mathcal{H}, V)$.

Note that strong and weak satisfiability in a one-dimensional Hilbert space $\mathcal{H}$ coincide, and we just say that φ is satisfiable in $\mathcal{H}$.

Properties of Quantum Logic

The following is (strongly) valid

- $p \leftrightarrow \neg\neg p$
- $p \vee \neg p$
- $(p \to q) \leftrightarrow (\neg q \to \neg p)$

The following is not always strongly true

- $p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r)$
- $p \vee (q \wedge r) \leftrightarrow (p \vee q) \wedge (p \vee q)$

- Unlike in intuitionistic logic, the negation in quantum logic is classical
- Unlike in classical logic, distributivity of ‘and’ over ‘or’ (and ‘or’ over ‘and’) does not always hold in quantum logic.

Decidability and Complexity

The strong (weak) *n*-dim satisfiability problem is to determine for a propositional Hilbert quantum logic formula φ and an *n*-dimensional Hilbert space $\mathcal{H}$ over a subfield of $\mathbb{C}$ whether φ is strongly (weakly) satisfiable in $\mathcal{H}$.

The strong and the weak *n*-dim satisfiability problems are decidable.

Theorem (Herrmann and Ziegler, 2011)

1. *The n-dimensional strong and weak satisfiability problems are NP-complete if $n = 1, 2$.*
   (same as with classical Boolean satisfiability)
2. *The n-dimensional strong and weak satisfiability problems are complete for the non-deterministic Blum-Shum-Smale model of computation if $n \geq 3$.*
   (random access to registers that contain real values)

Logic for Quantum Programs (on Hilbert spaces)

(Slightly simplified) Logic for Quantum Programs (LQP)

$$\varphi ::= p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid [\pi]\varphi$$
$$\pi ::= \varphi? \mid U \mid \pi^\dagger \mid \pi_1 \cup \pi_2 \mid \pi_1 ; \pi_2$$

where

- $p \in \mathrm{AtProp}$ is an atomic proposition symbols,
- $U \in \mathcal{U}$ is a unitary operator symbol.

This language is almost the same as propositional dynamic logic, and was developed in:

- A. Baltag and S. Smets. LQP: the dynamic logic of quantum information. *Mathematical Structures in Computer Science*, 16 (2006), 3, 491–525.

LQP Semantics structures

Definition (Hilbert realization of LQP)

A Hilbert realization for LQP is a tuple $(\mathcal{H}, V_p, V_u)$ where

1. $\mathcal{H}$ is a Hilbert space,
2. $V_p : \mathrm{AtProp} \to L(\mathcal{H})$
3. V_u maps unitary operator symbol U to a unitary operator on $\mathcal{H}$
(a unitary is a linear operator T, such that $T^{-1} = T^{\dagger}$)

For a subset $A \subseteq \mathcal{H}$ and vector v, let

- $\mathrm{ClSp}(A) = \mathrm{cl}(\mathrm{sp}(A))$ be the closure of the span of A.
- $\mathrm{Proj}_A\, v$ be the projection of v onto $\mathrm{ClSp}(A)$.

If x is a one-dimensional subspace,
$\mathrm{Proj}_A\, x = \{\mathrm{Proj}_A\, v \mid v \in x\}$ is a subspace.

Semantics

We interpret formulas as subsets (not necessarily closed subspaces) of a Hilbert space.

$$
\begin{array}{lcl}
[\![p]\!] & = & V_p(p) \\
[\![\neg\varphi]\!] & = & \mathcal{H} \setminus [\![\varphi]\!] \\
[\![\varphi_1 \wedge \varphi_2]\!] & = & [\![\varphi_1]\!] \cap [\![\varphi_2]\!] \\
[\![[\pi]\varphi]\!] & = & \{s \mid t \in [\![\varphi]\!] \text{ whenever } s[\![\pi]\!]t\} \\
\hline
[\![\varphi?]\!] & = & \{(s,t) \mid t = \mathrm{Proj}_{[\![\varphi]\!]}(s)\} \\
[\![U]\!] & = & V_u(U) \\
[\![\pi^\dagger]\!] & = & [\![\pi]\!]^\dagger \\
[\![\pi_1 \cup \pi_2]\!] & = & [\![\pi_1]\!] \cup [\![\pi_2]\!] \\
[\![\pi 1; \pi_2]\!] & = & \{(s,t) \mid \exists u, s[\![\pi_1]\!]u, u[\![\pi_2]\!]t\}
\end{array}
$$

Identify any set $X \cup \{0\}$ with the set $X \setminus \{0\}$.

Note that $\neg$ is interpreted using the set-theoretic complement (modulo 0), rather than orthocomplement.

Probabilistic Logic of Quantum Programs

Add to the Logic of Quantum Programs, formulas

$$P^{\geq r}\varphi$$

with semantic clause

$$[\![P^{\geq r}\varphi]\!] = \{0\} \cup \{s \neq 0 \mid \langle v, \mathrm{Proj}_{[\![\varphi]\!]}(v)\rangle \geq r \text{ where } v = s/\|s\|\}$$

or add linear combinations of probability formulas

$$a_1 P(\varphi_1) + \cdots + a_n P(\varphi_n) \geq r$$

with semantic clause

$$[\![a_1 P(\varphi_1) + \cdots + a_n P(\varphi_n) \geq r]\!]$$
$$= \{0\} \cup \{s \neq 0 \mid \sum_{k=1}^{n} a_k \langle v, \mathrm{Proj}_{[\![\varphi_k]\!]}(v)\rangle \geq r \text{ where } v = s/\|s\|\}$$

Abbreviations

$$\sim\varphi \quad = \quad [\varphi?]\bot \quad \text{(orthocomplement)}$$

Abbreviations without linear combinations

$$\begin{aligned}
P^{\leq r}\varphi &= P^{\geq 1-r}\sim\varphi \\
P^{<r}\varphi &= \neg P^{\geq r}\varphi \\
P^{>r}\varphi &= \neg P^{\leq r}\varphi \\
P^{=r}\varphi &= P^{\geq r}\varphi \wedge P^{\leq r}\varphi
\end{aligned}$$

Abbreviations with linear combinations

$$\begin{aligned}
a\sum_{k=1}^{n} a_k P(\varphi_k) &= \sum_{k=1}^{n} a \cdot a_k P(\varphi_k) \\
t \leq r &= -t \geq -r \\
t < r &= \neg(t \geq r) \\
t > r &= \neg(t \leq r) \\
t_1 \geq t_2 &= t_1 - t_2 \geq 0 \\
t_1 = t_2 &= t_1 \geq t_2 \wedge t_2 \geq t_1
\end{aligned}$$

Basic properties of quantum probability

1. $P(\varphi) + P(\neg\varphi) \neq 1$ for some φ
2. $P(\varphi) + P({\sim}\varphi) = 1$ for all φ

Compositing Systems

Definition (Tensor product of spaces)

Let V and W be Hilbert spaces with finite bases B and C. Then $V \otimes W$ is a Hilbert space with

1. basis $B \times C$ (Cartesian product of B and C). Each element of $B \times C$ is written $b \otimes c$ for $b \in B$ and $c \in C$
2. inner product $\langle \cdot, \cdot \rangle$ such that $\langle (b_1 \otimes c_1), (b_2 \otimes c_2) \rangle = \langle b_1, b_2 \rangle \langle c_1, c_2 \rangle$.

Entengled vs separable

Definition (Tensor product of states)

Let $\otimes : V \times W \to V \otimes W$ (not surjective), such that

- $(w + v) \otimes x) = (w \otimes x) + (v \otimes x)$
- $w \otimes (v + x) = (w \otimes v) + (w \otimes x)$
- $c(w \otimes v) = (cw) \otimes v = w \otimes (cv)$
- $b \otimes c = (b, c)$ for $b \in B, c \in C$
  [If $V \otimes W$ were defined by B and C]

$x \in V \otimes W$ is separable if there exist $v \in V, w \in W$, such that $x = v \otimes w$. Otherwise x is entangled.

Tensor product of linear operators

Let $T : V \to V$ and $R : W \to W$ be linear maps. Then $T \otimes R : V \otimes W \to V \otimes W$ by

$$(T \otimes R)(x \otimes y) = T(x) \otimes R(y)$$

If $B = (b_1, \ldots, b_m)$ is a basis for V and $C = (c_1, \ldots, c_n)$ is a basis for W, then

$$(T \otimes R)(\sum_{i=1,j=1}^{i=m,j=n} a_{i,j}(b_i \otimes c_j) = \sum_{i=1,j=1}^{i=m,j=n} a_{i,j}(T(b_i) \otimes R(c_j)).$$

A more complete LQP

A more complete Logic for Quantum Programs (Slightly simplified) Logic for Quantum Programs (LQP)

$$\varphi ::= \top_I \mid p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid [\pi]\varphi$$
$$\pi ::= K_I \mid \varphi? \mid U \mid \pi^\dagger \mid \pi_1 \cup \pi_2 \mid \pi_1 ; \pi_2$$

where

- $p \in \mathrm{AtProp}$ is an atomic proposition symbols,
- $U \in \mathcal{U}$ is a unitary operator symbol.
- $I \subseteq \{0, \dots, N-1\}$ is a subset of N agents.

This language is essentially the one developed in:

- A. Baltag and S. Smets. LQP: the dynamic logic of quantum information. *Mathematical Structures in Computer Science*, 16 (2006), 3, 491–525.

full LQP Semantics structures

Definition (Hilbert realization of full LQP)

A Hilbert realization for LQP is a tuple $(\mathcal{H}, V_i, V_p, V_u)$ where

1. $\mathcal{H}$ is a Hilbert space,
2. V_i maps each $j \in \{0, \ldots, N-1\}$ to a Hilbert space $\mathcal{H}_j$, such that $H = \bigotimes \mathcal{H}_j$.
3. $V_p : \text{AtProp} \to L(\mathcal{H})$
4. V_u maps unitary operator symbol U to a unitary operator on $\mathcal{H}$

Semantics

Identify N with $\{0, \ldots, N-1\}$.

$$
\begin{array}{lcl}
[\![p]\!] & = & V_p(p) \\
[\![\neg\varphi]\!] & = & \mathcal{H} \setminus [\![\varphi]\!] \\
[\![\varphi_1 \wedge \varphi_2]\!] & = & [\![\varphi_1]\!] \cap [\![\varphi_2]\!] \\
[\![[\pi]\varphi]\!] & = & \{s \mid t \in [\![\varphi]\!] \text{ whenever } s[\![\pi]\!]t\} \\
[\![\top_I]\!] & = & \{s \mid s = v \otimes w \text{ for some} \\
 & & v \in \bigotimes_{i \in I} \mathcal{H}_i, w \in \bigotimes_{j \in N \setminus I} \mathcal{H}_j\} \\
\hline
[\![K_I]\!] & = & \{(s,t) \in [\![\top_I]\!] \times [\![\top_{N\setminus I}]\!] \mid \text{ there is a unitary} \\
 & & U_{N\setminus I} \text{ over } \bigotimes_{j \in N\setminus I} \mathcal{H}_j \text{ such that} \\
 & & t = (Id_I \otimes U_{N\setminus I})(s)\} \\
[\![\varphi?]\!] & = & \{(s,t) \mid t = \{\mathrm{Proj}_{[\![\varphi]\!]}(v) \mid v \in s\} \\
[\![U]\!] & = & V_u(U) \\
[\![\pi^\dagger]\!] & = & [\![\pi]\!]^\dagger \\
[\![\pi_1 \cup \pi_2]\!] & = & [\![\pi_1]\!] \cup [\![\pi_2]\!] \\
[\![\pi 1; \pi_2]\!] & = & \{(s,t) \mid \exists u, s[\![\pi_1]\!]u, u[\![\pi_2]\!]t\}
\end{array}
$$

Epistemic operators

The operators K_I act as epistemic operators for the agents in I.

The idea behind a minimal quantum logic

Two characteristic properties of a Hilbert lattice is that

1. The lattice has a complement that behaves like classical negation
2. The lattice is not distributive

There are many more properties, but the simplest lattice structure considered to be relevant to a quantum setting is an ortholattice, a lattice with a well-behaved complement.

Ortholattice

Definition (Ortholattice)

An orthalattice is a tuple $\mathbb{L} = (L, \leq, (\cdot)')$, such that

1. $\mathbb{L}$ is bounded: there exists a smallest element $\mathbf{0}$ and a largest element $\mathbf{1}$
2. $a \wedge a' = \mathbf{0}$ and $a \vee a' = \mathbf{1}$ for each $a \in L$
3. $a = (a')'$
4. $a \leq b$ if and only if $b' \leq a'$.

Related work

- M. Dalla Chiara, R. Giuntini, and R. Greechie. *Reasoning in Quantum Theory: Sharp and unsharp quantum logics.* Kluwer academic publisher, 2004.
- R.I. Goldblatt. Semantic Analysis of Orthologic. *Journal of Philosophical Logic* **3**(1/2): 19–35, 1974.

Ortholattice Realization of Orthologic

The Minimal Quantum Logic is called Orthologic, and uses the classical propositional language.

Definition (Ortholattice realization of orthologic)

An ortholattice realization of orthologic is a pair $\mathbb{L} = (L, V)$ where $L = (A, \leq, (\cdot)')$ is an ortholattice and $V : \text{AtProp} \to L$ is valuation function mapping atomic proposition letters to elements of the lattice.

V extends to all formulas as follows:

1. $V(\neg\varphi) = V(\varphi)'$
2. $V(\varphi \wedge \psi) = V(\varphi) \wedge V(\psi)$.

We write $\mathbb{L} \models \varphi$ if $\mathbb{L} = (L, V)$ and $V(\varphi) = \mathbf{1}$.

Orthoframe

Definition

An orthogonality orthoframe is a tuple $(X, \perp)$, such that X is a set, and $\perp \subseteq X \times X$ is a relation satisfying

1. for no a does it hold that $a \perp a$ ($\perp$ is irreflexive)
2. if $a \perp b$ then $b \perp a$ ($\perp$ is symmetric)

Definition

A non-orthogonality orthoframe is a tuple $(X, \not\perp)$, such that X is a set, and $\not\perp \subseteq X \times X$ is a relation satisfying

1. $a \not\perp a$ for each $a \in A$ ($\not\perp$ is reflexive)
2. if $a \not\perp b$ then $b \not\perp a$ ($\not\perp$ is symmetric)

- Given an orthogonality orthoframe $(X, \perp)$, $(X, X \times X \setminus \perp)$ is a non-orthogonality orhoframe.
- Given a non-orthogonality orthoframe $(X, \not\perp)$, $(X, X \times X \setminus \not\perp)$ is an orthogonality orthoframe.

Bi-orthogonal closure of in an orthoframe

For $S, T \subseteq X$ and $a \in X$, let

- $a \perp S$ iff $a \perp b$ for every $b \in S$, and let
- $S \perp a$ iff $S \perp a$.
- $S \perp T$ iff $a \perp T$ for every $a \in S$.
- $S^{\perp} = \{a \mid a \perp S\}$

Definition (Bi-orthogonally closed)

A set $S \subseteq X$ is called (bi-orthogonally) closed in an orthoframe $F = (X, \not\perp)$ if

$$S = (S^{\perp})^{\perp}.$$

Orthoframe realization of orthologic

Definition

An *orthoframe realization* of orthologic is a tuple $(X, \not\perp, P, V)$, where

1. $(X, \not\perp)$ is an orthoframe
2. $P \subseteq \mathcal{P}(X)$ consists of bi-orthogonally closed sets, includes X, $\emptyset$, and is closed under orthocomplement $\perp$ and set theoretic intersection $\cap$.
3. V : AtProp $\rightarrow P$ is a valuation function.

V can be extended to all formulas by:

1. $V(\neg\varphi) := V(\varphi)^{\perp}$.
2. $V(\varphi \wedge \psi) = V(\varphi) \cap V(\psi)$.

We write $\mathbb{F} \models \varphi$ if $\mathbb{F} = (X, \not\perp, P, V)$ and $V(\varphi) = X$.

ortholattice to orthoframe

Given an lattice realization of orthologic $\mathbb{L} = (A, \leq, -, V^L)$, let $K^{\mathbb{L}} = (X, \not\perp, P, V^K)$ be given by

1. $X = A \setminus \{\mathbf{0}\}$.
2. $a \not\perp b$ iff $a \not\leq -b$
3. $P = \{\{x \in X \mid x \leq a\} \mid a \in A\}$
4. $V^K(p) = \{b \in X \mid b \leq V^L(p)\}$

Then

1. $K^{\mathbb{L}}$ is an orthoframe realization of orthologic
2. for every φ, $\mathbb{L} \models \varphi$ if and only if $K^{\mathbb{L}} \models \varphi$

orthoframe to ortholattice

Given an orthoframe realization $\mathbb{K} = (X, \not\perp, P, V^K)$, let $L^{\mathbb{K}} = (A, \leq, -, V^L)$ be given by

1. $A := P$
2. $a \leq b$ iff $a \subseteq b$ for each $a, b \in A$
3. $-a := \{b \in X \mid a \perp b\}$
4. $V^L(p) := V^K(p)$.

Then

1. $L^{\mathbb{K}}$ is an ortholattice realization of orthologic
2. for every φ, $\mathbb{K} \models \varphi$ if and only if $L^{\mathbb{K}} \models \varphi$.

Connection to modal logic

We consider the following basic modal language

$$\varphi ::= p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Box\varphi$$

Let $\Diamond\varphi = \neg\Box\neg\varphi$.
We are interested in the system B (named after Brouwer) with axiom

$$\varphi \to \Box\Diamond\varphi$$

This axiom corresponds to Kripke frames being symmetric.
Define $\Box : \mathcal{P}(X) \to \mathcal{P}(X)$ by

$$\Box A = \{x \in X \mid \forall y,\ x \not\perp y \Rightarrow y \in A\}.$$

B-frame realization

A B-realization is a tuple $(X, \not\perp, P, V)$, such that

- $(X, \not\perp)$ is an orthoframe
- $P \subseteq \mathcal{P}(X)$ consists of $\emptyset, X$ and is closed under set-complement $-$, intersection $\cap$, and the modal operator $\Box$
- $V : \text{AtProp} \to P$

V extends to all formulas as follows:

- $V(\neg\varphi) = X - V(\varphi)$
- $V(\varphi \wedge \psi) = V(\varphi) \cap V(\psi)$
- $V(\Box\varphi) = \Box V(\varphi)$

Write $x \models \varphi$ if $x \in V(\varphi)$.

Orthologic to modal logic

Define function τ from the propositional logic language to the basic modal language as follows

1. $\tau(p) = \Box\Diamond p$
   The addition of $\Box\Diamond$ is to reestablish bi-orthogonal closure.
2. $\tau(\neg\varphi) = \Box\neg\tau(\varphi)$
   Negation on the left is an orthogonal complement, and on the right it is a set-complement
3. $\tau(\varphi \wedge \psi) = \tau(\varphi) \wedge \tau(\psi)$.

ortholattice to B-frame

Given an ortholattice realization $\mathbb{O} = (X, \not\perp, P^o, V)$, let $B^{\mathbb{O}} = (X, \not\perp, P^b, V)$, such that

- P^b is the smallest set containing $V(p)$ for each $p \in \mathrm{AtProp}$ and is closed under set-complement $-$, intersection $\cap$, and the modal operator $\Box$.

Then

1. $B^{\mathbb{O}}$ is a B-realization
2. for every x and φ,

$$\mathbb{O}, x \models_{OL} \varphi \Leftrightarrow B^{\mathbb{O}}, x \models_{B} \tau(\varphi)$$

B-frame to ortholattice

Given a B-realization $\mathbb{B} = (X, \not\perp, P^b, V^b)$, let $O^{\mathbb{B}} = (X, \not\perp, P^o, V^o)$, such that

- P^o is the smallest set containing $V^b(\Box\Diamond p)$ for each $p \in \text{AtProp}$ and is closed under orthocomplement $\perp$ and intersection $\cap$.
- $V^o(p) = V^b(\Box\Diamond p)$

Then

1. $B^{\mathbb{O}}$ is an orthoframe realization
2. for every x and ψ,

$$O^{\mathbb{B}}, x \models_{OL} \varphi \Leftrightarrow \mathbb{B}, x \models_B \tau(\varphi)$$

Axiomatization: Rules involving conjunction

$$T \cup \{\varphi\} \vdash \varphi \qquad \text{(Identity)}$$

$$\frac{T \vdash \varphi, R \cup \{\varphi\} \vdash \psi}{T \cup R \vdash \psi} \qquad \text{(Transitivity)}$$

$$T \cup \{\varphi \wedge \psi\} \vdash \varphi \qquad (\wedge\text{-elimination})$$

$$T \cup \{\varphi \wedge \psi\} \vdash \psi \qquad (\wedge\text{-elimination})$$

$$\frac{T \vdash \varphi, T \vdash \psi}{T \vdash \varphi \wedge \psi} \qquad (\wedge\text{-introduction})$$

$$\frac{T \cup \{\varphi, \psi\} \vdash \chi}{T \cup \{\varphi \wedge \psi\} \vdash \chi} \qquad (\wedge\text{-introduction})$$

Axiomatization: Rules involving negation

$$T \cup \{\varphi\} \vdash \neg\neg\varphi \qquad \text{(double negation)}$$

$$T \cup \{\neg\neg\varphi\} \vdash \varphi \qquad \text{(double negation)}$$

$$\frac{\{\varphi\} \vdash \psi, \{\varphi\} \vdash \neg\psi}{\emptyset \vdash \neg\varphi} \qquad \text{(absurdity)}$$

$$T \cup \{\varphi \wedge \neg\varphi\} \vdash \psi \qquad \text{(contradiction)}$$

$$\frac{\{\varphi\} \vdash \psi}{\{\neg\psi\} \vdash \neg\varphi} \qquad \text{(contrapositive)}$$

Concepts about this proof system

Definition (Consistent)

T is inconsistent if $T \vdash \varphi \wedge \neg\varphi$ for same φ, and is consistent otherwise.

Definition (Deductively Closed)

T is deductively closed if $\{\varphi \mid T \vdash \varphi\} \subseteq T$.

Lemma (Weak Lindenbaum)

If $T \nvdash \neg\varphi$, then there is a set S, such that

1. *for all ψ, $T \vdash \psi \Rightarrow S \nvdash \neg\psi$ (compatability) and*
2. $S \vdash \varphi$

Canonical Model

Let $\mathbb{K} = (X, \not\perp, P, V)$ be an alleged canonical model, where

1. X is the set of all consistent deductively closed sets of formulas.
2. $T_1 \not\perp T_2$ iff for all φ, $T_1 \vdash \varphi$ implies $T_2 \nvdash \neg\varphi$.
3. P is the collection of sets $\mathcal{S} \subseteq X$, such that

$$T \in \mathcal{S} \Leftrightarrow [\forall U \in X((T \not\perp U) \Rightarrow \exists V(U \not\perp V \ \& \ V \in \mathcal{S}))]$$

4. $V(p) = \{T \in X \mid p \in T\}$

Show that $\mathbb{K}$ is a realization

We first show that $\not\perp$ is reflexive and symmetric

- $\not\perp$ is reflexive, since each $T \in X$ is consistent.
- $\not\perp$ is symmetric, since:
  - If $T_1 \vdash \varphi \Rightarrow T_2 \nvdash \neg\varphi$
  - Then $T_1 \vdash \neg\varphi \Rightarrow T_2 \nvdash \neg\neg\varphi \Rightarrow T_2 \nvdash \varphi$

P is bi-orthogonally closed (an exercise)
V : AtProp $\to P$ by the weak Lindenbaum lemma

Truth Lemma

Lemma (Truth Lemma)

For any $T \in X$ and formula φ,

$$T \models \varphi \Leftrightarrow \varphi \in T$$

This is proved by induction on the structure of φ. The negation case uses the weak Lindenbaum lemma.

Completeness

We show contra positively

$$T \nvdash \varphi \Rightarrow T \not\models \varphi$$

1. If $T \nvdash \varphi$ then T is consistent
2. Let Z be the deductive closure of T (Hence Z is in X).
3. Then $Z \models T$ (by truth lemma)
4. But $Z \not\models \varphi$ (otherwise $\varphi \in Z$ and $T \vdash \varphi$)

Modularity and Orthomodularity

Definition (Modular lattice)

A modular lattice is a lattice $(A, \leq)$ that satisfies the following modular law:

$$a \leq b \Rightarrow \forall c, a \vee (c \wedge b) = (a \vee c) \wedge b.$$

Definition (Orthomodular lattice)

An orthomodular lattice is an ortholattice $(A, \leq, (\cdot)')$ that satisfied the following orthomodular law:

$$a \leq b \Rightarrow b \wedge (b' \vee a) = a$$

Relationship between modularity and orthomodularity

- Both modularity and orthomodularity are weak versions of distributivity.
- A modular ortholattice is always an orthomodular lattice
- The closed linear subspaces of a Hilbert space form an orthomodular lattice
- The closed linear subspaces of a Hilbert space form a modular ortholattice if and only if the Hilbert space is finite

Orthomodularity is sometimes called weak modularity

Equivalent characterizations of orthomodularity

The following are equivalent

1. $a \leq b$ implies $b \wedge (b' \vee a) = a$ (definition),
2. $a \leq b$ implies $a \vee (a' \wedge b) = b$,
3. $a \wedge (a' \vee (a \wedge b)) \leq b$.
4. $a \leq b$ if and only if $a \wedge (a \wedge b)' = \mathbf{0}$
5. ($a \leq b$ and $b \wedge a' = \mathbf{0}$) implies $a = b$

The idea behind Orthomodular Quantum Logic

1. Hilbert lattices are orthomodular.
2. Adding orthomodularity to the framework is relatively straightforward
3. Adding orthomodularity adds a degree of distributivity to the framework that is useful

There are still more properties of Hilbert lattices, but orthomodularity stands out as relevant to a quantum setting.

The language of orthomodular quantum logic is just the same as for classical propositional logic.

Realizations for orthomodular quantum logic

An algebraic realization of orthomodular quantum logic is a pair $\mathbb{L} = (L, V)$ such that $\mathbb{L}$ is an ortholattice realization of orthologic and L is an orthomodular lattice.

A Kripkean realization of orthomodular quantum logic is a tuple $\mathbb{K} = (X, \not\perp, P, V)$, such that $\mathbb{K}$ is a orthoframe realization of orthologic and for every $a, b \in P$,

$$a \not\subseteq b \Rightarrow a \cap (a \cap b)^{\perp} \neq \emptyset.$$

orthomodular lattice to orthomodular orthoframe

Recall the transition from a lattice realization of orthologic $\mathbb{L} = (A, \leq, -, V^L)$, to an orthoframe realization $K^{\mathbb{L}} = (X, \not\perp, P, V^K)$ be given by

1. $X = A \setminus \{\mathbf{0}\}$.
2. $a \not\perp b$ iff $a \not\leq -b$
3. $P = \{\{x \in X \mid x \leq a\} \mid a \in A\}$
4. $V^K(p) = \{b \in X \mid b \leq V^L(p)\}$

Then

1. If $\mathbb{L}$ is an algebraic realization of orthomodular quantum logic, then $K^{\mathbb{L}}$ is a Kripkean realization of orthomodular quantum logic.

orthomodular orthoframe to orthomodular lattice

Recall the translation of an orthoframe realization $\mathbb{K} = (X, \not\perp, P, V^K)$ to an ortholattice realization $L^{\mathbb{K}} = (A, \leq, -, V^L)$ be given by

1. $A := P$
2. $a \leq b$ iff $a \subseteq b$ for each $a, b \in A$
3. $-a := \{b \in X \mid a \perp b\}$
4. $V^L(p) := V^K(p)$.

Then

1. If $\mathbb{K}$ is a Kripkean realization of orthomodular quantum logic, then $L^{\mathbb{K}}$ is an algebraic realization of orthomodular quantum logic.

Axiomatization of orthomodular quantum logic

An axiomatization of orthomodular quantum logic consists of the rules for orthologic together with the following rule:

$$\varphi \wedge (\neg\varphi \vee (\varphi \wedge \psi)) \vdash \psi$$

The proof of soundness is straightforward, but the proof of completeness needs a slight modification to the canonical model construction of the set P:

- P is the set of collection of sets $\mathcal{S} \in X$, such that

  $$T \in \mathcal{S} \Leftrightarrow [\forall U \in X((T \not\perp U) \Rightarrow \exists V(U \not\perp V \;\&\; V \in \mathcal{S}))]$$

  and $\mathcal{S} = V(\varphi)$ for some φ.

Toward characterizing Hilbert lattices

- D. Aerts, Quantum axiomatics. In Kurt Engesser, Dov M. Gabbay, and Daniel Lehmann, (eds.), *Handbook of Quantum Logic and Quantum Structures*, 1st edn., Elsevier Science B.V., Amsterdam, 2009, pp. 79–126.
- R. Mayet. Some characterizations of the underlying division ring of a Hilbert lattice by automorphisms. *International Journal of Theoretical Physics*, 37, 109–114, 1998.
- C. Piron. *Foundations of Quantum Physics.* W.A. Benjamin, Inc. 1976.

Complete and atomic lattices

Definition (Complete lattice)

A lattice L is complete if for any $A \subseteq L$, its meet $\bigwedge A$ and join $\bigvee A$ are in L.

- For $a, b \in L$, we say that b covers a if $a < b$ and if $a \leq c < b$ then $a = c$.
- Call $a \in L$ an atom if a covers $\mathbf{0}$

Definition (Atomic lattice)

A lattice is atomistic if for any $p \neq \mathbf{0}$, there is an atom a such that $a \leq p$.

A lattice is atomistic if every $p > 0$ is the join of atoms.

A complete orthomodular lattice is atomic if and only if it is atomistic.

Covering Law

Definition (Covering law)

A lattice satisfies the covering law if whenever a an atom and $a \wedge b = \mathbf{0}$, then $a \vee b$ covers b.

An equivalent characterization of the covering law in an orthomodular lattice is

A lattice satisfies the covering law if whenever a is an atom and $a \not\leq p'$ then $p \wedge (p' \vee a)$ is an atom.

The Sasaki projection is defined by

$$p[a] = p \wedge (p' \vee a)$$

When $p \not\perp a$, think of $p[a]$ as the atom under p to a.

Propositional System

Definition (Propositional system)

A propositional system is an orthomodular lattice that

1. is complete (Contains arbitrary meets and joints)
2. is atomic (Every non-zero element is above an atom)
3. satisfies the covering law
   (If a is an atom a and $a \not\leq p$, then $p[a]$ is at atom)

Irreducibility and superposition

A direct union between involuted lattice $\mathbb{L}_j = (L_j, \leq_j -_j)$ $(j \in \{1, 2\})$ is the lattice $\mathbb{L} = (L, \leq, -)$, where

1. $L = L_1 \times L_2$
2. $(a_1, b_1) \leq (a_2, b_2)$ if and only if $a_i \leq b_i$ for each $i \in \{1, 2\}$.
3. $-(a, b) = (-a, -b)$.

Definition (Irreducible)

A lattice is irreducible if it is not the direct sum of two lattice each with at least two elements.

In a propositional system, irreducibility is equivalent to the following

Definition (Superposition Principle)

For any two distinct atoms a, b, there is an atom c, distinct from both a and b, such that $a \vee c = b \vee c = a \vee b$.

Piron lattice

Definition (Piron lattice)

A Piron lattice is a propositional system that satisfies the superposition principle.

Satisfying the superposition principle essentially states that the lattice is reasonably well connected (this will be easier to see when we look at frames).

Theorem

A Piron lattice with at least four orthogonal points is isomorphic to the lattice of bi-orthogonally closed subspaces of a generalized Hilbert space.

What is a generalized Hilbert space?

What is a generalized Hilbert space?

A generalized Hilbert space over a division ring K is a tuple $(V, (\cdot)^\star, \langle \cdot, \cdot \rangle)$, such that

1. V is a module over K (a vector space of a ring)
2. $(\cdot)^\star : K \to K$ is an involution, and hence has properties
   1. $(v^\star)^\star = v$
   2. $(vw)^\star = w^\star v^\star$
3. $\langle \cdot, \cdot \rangle : V \times V \to K$ is such that for all $v, w, x \in V$, $S \subseteq V$, and $a \in K$
   1. $\langle x, av + w \rangle = \langle x, v \rangle + a \langle x, w \rangle$
   2. $\langle x, y \rangle = \langle y, x \rangle^\star$
   3. $\langle x, x \rangle = 0$ iff $x = 0$
   4. $M^\perp + (M^\perp)^\perp = V$, where $M^\perp = \{y \in V \mid \langle y, x \rangle \forall x \in M\}$

A generalized Hilbert space is also called an orthomodular vector space.

Mayet's condition

An ortholattice isomorphism is a bijective function between two lattices that preserves meets and orthocomplement.

Definition (Mayet's condition)

A Piron lattice satisfies Mayet's condition if there is an automorphism $k : L \to L$ such that

1. there is a $p \in L$ such that $k(p) < p$, and
2. there is a $q \in L$ such that there are at least two distinct atoms below q and $k(r) = r$ for all $r \leq q$.

Theorem

An orthomodular vector space is an infinite dimensional Hilbert space over the complex numbers, real numbers, or quaternions if and only if its lattice of bi-orthogonally closed subspaces is a Piron lattice that satisfies Mayet's condition.

Is there a Kripke frame characterization?

- A. Baltag and S. Smets. Complete axiomatizations for quantum actions. International Journal of Theoretical Physics 44: 2267–2282, 2005.
- Jort Bergfeld, Kohei Kishida, Joshua Sack, and Shengyang Zhong. Duality for the Logic of Quantum Actions. Published, online first, in *Studia Logica*, November 2014.

Dynamic frame

Definition (Dynamic frame)

A dynamic frame is a tuple $(\Sigma, \mathcal{L}, \{\xrightarrow{P?}\}_{P\in\mathcal{L}})$, where

1. Σ is a set
2. $\mathcal{L} \subseteq \mathcal{P}(\Sigma)$ is closed under intersection $\cap$ and orthocomplement $(\cdot)^\perp : A \mapsto \{y \in \Sigma \mid x \perp y,\ \forall x \in A\}$
3. $\xrightarrow{P?} \subseteq \Sigma \times \Sigma$ (let $\not\perp = \bigcup_{P\in\mathcal{L}} \xrightarrow{P?}$)

A Hilbert space $\mathcal{H}$ gives rise to a dynamic frame where

1. Σ consists of one-dimensional subspaces
2. $\mathcal{L}$ is the lattice of closed linear subspaces
3. $s \xrightarrow{P?} t$ iff the projection of s onto P in $\mathcal{H}$ is t.

Dynamic frames with certain constraints are dual to Piron lattices.

Basic properties

Definition (Atomicy)

A dynamic frame satisfies atomicity if for any $s \in \Sigma$, $\{s\} \in \mathcal{L}$.

Each state is an atom.

Definition (Adequacy)

A dynamic frame satisfies adequacy if for any $s \in \Sigma$ and $P \in \mathcal{L}$, if $s \in P$, then $s \xrightarrow{P?} s$.

If $\xrightarrow{P?}$ were a partial function, it would fix P.

Definition (Repeatability)

A dynamic frame satisfies repeatability if any $s, t \in \Sigma$ and $P \in \mathcal{L}$, if $s \xrightarrow{P?} t$, then $t \in P$.

If $\xrightarrow{P?}$ were a partial function, its image would be in P.

Self-Adjointness

Definition (Self-Adjointness)

A dynamic frame satisfies self-adjointness if for any $s, t, u \in \Sigma$ and $P \in \mathcal{L}$, if $s \xrightarrow{P?} t \not\perp u$, then there is a $v \in \Sigma$ such that $u \xrightarrow{P?} v \not\perp s$.

Covering property

Definition (Covering property)

A dynamic frame satisfies the covering property if

- when $s \xrightarrow{P?} t$ for $s, t \in \Sigma$ and $P \in \mathcal{L}$,
- Then, for any $u \in P$, if $u \neq t$ then $u \to v \not\to s$ for some $v \in P$

Contrapositively, $u = t$ if $u \to v$ implies $v \to s$ for all $v \in P$.

The covering property (together with other properties) gives t a unique property ($u \to v$ implies $v \to s$), and hence makes $\xrightarrow{P?}$ a partial function.

Superposition

Definition (Proper superposition)

A dynamic frame satisfies proper superposition if for any $s, t \in \Sigma$ there is a $u \in \Sigma$ such that $s \to u \to t$.

This means

- Any two states can be reached via two non-orthogonality steps.
- The composition of non-orthogonality with itself is the total relation, and its modality is the universal modality

Quantum Dynamic Frame

Definition (Quantum Dynamic Frame)

A dynamic frame is a quantum dynamic frame if it satisfies

1. Atomicity
2. Adequacy
3. Repeatability
4. Self-adjointness
5. Covering property
6. Proper superposition

Piron lattice to Quantum Dynamic Frame

Given a Piron lattice $\mathfrak{L} = (L, \leq, -)$, let $F(\mathfrak{L}) = (\Sigma, \mathcal{L}, \{\xrightarrow{P?}\}_{P \in \mathcal{L}})$ be defined by

1. Σ is the set of atoms of $\mathfrak{L}$
2. $\mathcal{L}$ is the set $\{\{a \mid a \leq p, a \text{ is an atom}\} \mid p \in L\}$.
3. For each $x \in \mathcal{L}$, where $p = \bigvee x$, define $\xrightarrow{x?} \subseteq \Sigma \times \Sigma$ by $a \xrightarrow{x?} b$ if and only if $p \wedge (-p \vee a) = b$.

Then $F(\mathfrak{L})$ is a quantum dynamic frame

Quantum Dynamic Frame to Piron lattice

Given a quantum dynamic frame $\mathfrak{F} = (\Sigma, \mathcal{L}, \{\xrightarrow{P?}\}_{P \in \mathcal{L}})$, let $G(\mathfrak{F}) = (\mathcal{L}, \subseteq, \sim)$, where

- $\sim A = \{s \mid s \not\to t, \forall t \in A\}$.

Then $G(\mathfrak{F})$ is a Piron lattice

Frame isomorphisms

Definition (Quanum dynamic frame isomorphism)

A function $f : \Sigma_1 \to \Sigma_2$ is a quantum dynamic frame isomorphism from $(\Sigma_1, \mathcal{L}_1, \{\xrightarrow{P?}_1\}_{P \in \mathcal{L}_1})$ to $(\Sigma_2, \mathcal{L}_2, \{\xrightarrow{P?}_2\}_{P \in \mathcal{L}_2})$ if

1. f is a bijection
2. $s \not\perp t$ if and only if $f(s) \not\perp f(t)$ for each $s, t \in \Sigma_1$.

Logic for Quantum Programs (on Frames)

Recall the language of the Logic for Quantum Programs

$$\varphi ::= p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid [\pi]\varphi$$
$$\pi ::= \varphi? \mid U \mid \pi^{\dagger} \mid \pi_1 \cup \pi_2 \mid \pi_1 ; \pi_2$$

where

- $p \in \mathrm{AtProp}$ is an atomic proposition symbols,
- $U \in \mathcal{U}$ is a unitary operator symbol.

We we see how to interpret this language directly on frames.

LQP Semantics structures on frames

Definition (Frame realization of LQP)

A Frame realization for LQP is a tuple $(\mathfrak{F}, V_p, V_u)$ where

1. $\mathfrak{F} = (\Sigma, \mathcal{L}, \{\xrightarrow{P?}\}_{P \in \mathcal{L}})$ is a quantum dynamic frame,
2. V_p : AtProp $\rightarrow \mathcal{L}$
3. V_u maps unitary operator symbol U to an automorphism of $\mathfrak{F}$

Semantics

$$
\begin{array}{|lcl|}
\hline
[\![p]\!] & = & V_p(p) \\
[\![\neg\varphi]\!] & = & \Sigma \setminus [\![\varphi]\!] \\
[\![\varphi_1 \wedge \varphi_2]\!] & = & [\![\varphi_1]\!] \cap [\![\varphi_2]\!] \\
[\![[\pi]\varphi]\!] & = & \{s \mid t \in [\![\varphi]\!] \text{ whenever } s[\![\pi]\!]t\} \\
\hline
[\![\varphi?]\!] & = & \xrightarrow{P?} \text{ where } P = ([\![\varphi]\!]^{\perp})^{\perp} \\
[\![U]\!] & = & V_u(U) \\
[\![\pi^{\dagger}]\!] & = & [\![\pi]\!]^{-1} \\
[\![\pi_1 \cup \pi_2]\!] & = & [\![\pi_1]\!] \cup [\![\pi_2]\!] \\
[\![\pi 1; \pi_2]\!] & = & \{(s,t) \mid \exists u, s[\![\pi_1]\!]u, u[\![\pi_2]\!]t\} \\
\hline
\end{array}
$$