Understanding reasoning in games using utility proportional beliefs

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Abstract

Traditionally very little attention has been paid to the reasoning process that underlies a game theoretic solution concept. When modeling bounded rationality in one-shot games, however, the reasoning process can be a great source of insight. The reasoning process itself can provide testable assertions, which provide more insight than the fit to experimental data. Based on Bach and Perea's (2014) concept of utility proportional beliefs, I analyze the players' reasoning process and find three testable implications: (1) Players form an initial belief that is the basis for further reasoning. (2) Players reason by alternatingly considering their own and their opponent's incentives. (3) Players perform only several rounds of deliberate reasoning.

Keywords: Epistemic game theory, interactive epistemology, solution concepts, bounded rationality, utility proportional beliefs, reasoning.

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1 Introduction

Most of the ongoing research in game theory focuses on the prediction of players' choices. In this paper, I want to take another perspective and look at the players' reasoning process rather than their choices. This perspective can be especially helpful to understand experimental data from one-shot games without opportunities for learning or coordinating. Existing bounded rationality concepts (e.g. Quantal Response Equilibrium (QRE) by McKelvey and Palfrey (1995) or Cognitive Hierarchy Models (CHM) by Camerer et al. (2004)) focus mainly on the prediction of empirical frequencies rather than mimicking players' reasoning process accurately. Therefore, these concepts do not provide a clear rationale for players selecting certain choices. The basic idea of a concept might give a hint but not a clear insight. Most often, these models are evaluated by comparing their fit to the data. This method, however, tells us little about the validity of certain features of the models. A deeper insight into the reasoning process, on the other hand, could help to better understand a concept's characteristics and therefore which features work well and which do not. A concept that makes clear assertions about the reasoning process can be tested much more rigorously. In fact, a good concept should present clear assertions about the reasoning process that can be tested individually and in their interaction with each other. In this paper, I discuss a solution concept that is based on a general idea and provides detailed assertions about the reasoning process that can be tested individually.

Bach and Perea (2014), henceforth BP, suggest a concept for bounded rationality that builds up on a simple idea: the differences in probabilities a player assigns to his opponent's choices should be equal to the differences in the opponent's utilities for these choices. Consider the game in Figure 1.1. Iterated elimination of strictly dominated strategies and therefore also Nash equilibrium predict that the *Column Player* will choose *left* and therefore the *Row Player* will play *top*. If the *Row Player* doubts, however, that the *Column Player* is fully rational and therefore also deems middle sufficiently likely, *bottom* seems to be a more preferred choice. This way of thinking is quite common in real life situations. Utility proportional beliefs captures the idea of this tradeoff by the fact that the probabilities assigned to the choices have to be proportional to their utility. Following this idea, the *Row Player* would assign high probabilities to the *Column Player*'s choices *left* and *middle* because the difference in utilities for the two choices is rather small compared to the difference with the utility induced by *right*. Hence, it would be optimal for her to play *bottom*. Based on this simple idea, BP build a solution concept that does not rely on an equilibrium assumption and that is meant to take place in a single player's mind. For the two-player case, it turns out that the beliefs players can hold under the concept of utility proportional beliefs are unique. Since the uniqueness of beliefs simplifies the analysis of the reasoning process drastically, I will only consider the two-player case.

I develop an explicit formula that calculates the beliefs, which players can hold, directly. The explicit formula allows us to investigate the reasoning process of players with utility proportional beliefs thoroughly. Moreover, I can state clear assertions about the reasoning process.

		Column Player		
		left	*middle*	right
Row Player	*top*	5, 5	0, 4	2, 1
	bottom	4, 5	4, 4	1, 2

Figure 1.1: Asymmetric matching pennies

System 1	System 2
associative	rule-based
holistic	analytic
relatively undemanding of cognitive capacity	demanding of cognitive capacity
relatively fast	relatively slow
acquisition by biology, exposure, and personal experience	acquisition by cultural and formal tuition

Table 1: Properties of System 1 & 2

I find that the reasoning process of players, who hold utility proportional beliefs, resembles some intuitive properties. (1) Initial Belief: The players' reasoning process starts with an initial belief about the opponent, which is the basis for the further belief formation process. It considers the goodness of the opponent's choices according to her average utility. (2) Interactive Reasoning Procedure: Within the reasoning process players' change perspectives within every reasoning step. Hence, they consider their opponent's incentive structure and their own alternatingly. Every reasoning step starts with taking the opponent's perspective. First one forms a belief about one's own choice given the belief that one entertained about the opponent in the previous reasoning step. Then one takes the new belief of the opponent about oneself and forms a new belief about the opponent. The formation of the respective beliefs follows the same principle as the formation of the initial belief. Only that players do not consider the average utility but the expected utility given the opponent's belief from the previous reasoning step. (3) Finite Steps of Reasoning: In theory players forming utility proportional beliefs undertake infinite steps of reasoning. Their final beliefs, however, can be approximated with only a few steps of reasoning. Under the assumption that human beings cannot process small differences in probabilities, I find that most reasoning processes stop after a few steps.

These features correspond closely to findings in the psychology literature. In his book "Thinking, Fast and Slow" Kahneman (2011) advocates the idea of reasoning in two distinct ways. He calls the two modes of thinking System 1 and System 2, according to Stanovich et al. (2000). Note that the word system should not indicate an actual system but only serves as label for different modes of thinking. Stanovich et al. (2000) describe System 1 as an automatic and mostly unconscious way of thinking that demands little computational capacity. System 2 describes the idea of deliberate reasoning. It comes into play when controlled analytical thinking is needed. Table 1 summarizes the properties of the two systems according to Stanovich et al. (2000).

The idea of System 1 describes the unconscious first reaction to a situation, which happens almost immediately and without demanding a lot of cognitive energy. Moreover, Kahneman (2011) argues that the beliefs formed by System 1 are the basis for conscious reasoning within System 2. This is consistent with the initial belief in property (1). As I will show, the initial belief does not take into account any strategic interaction. It can be seen as an automatic initial reaction to the game. The deliberate reasoning process described in property (2) can be imagined as being executed by System 2 using the findings of System 1, or in this case the initial belief. Taking another player's perspective takes deliberate reasoning and can hardly be done automatically. Think of a chess game, one has an intuitive impression of how a next move could look like. To check whether that intuition is correct, requires the effort to take the other player's perspective and check for her reaction. This needs not only to be done for one move into the future but for several. Property (3), finite steps of reasoning, is closely related to the problem of limited working memory. Baddeley (1992) defines working memory as "... [A] brain system that provides temporary storage and manipulation of the information necessary for such complex cognitive tasks as language comprehension, learning, and reasoning." Since working memory is critical for reasoning and limited, human beings can only perform a limited number of reasoning without the help of tools. Therefore, a model resembling human reasoning should not predict an infinite amount of reasoning steps.

In Section 2 I will introduce the concept of utility proportional beliefs formally, following the notation of BP. BP introduce an algorithm that iteratively deletes those beliefs that cannot be hold under common belief in utility proportional beliefs. In Section 3, I will discuss how to obtain the beliefs players can hold under utility proportional beliefs without using BP's algorithm. Using an explicit formula for the beliefs, I investigate how players holding utility proportional beliefs reason. In Section 4 I will provide some examples to give an intuitive understanding of the reasoning process.

2 Utility proportional beliefs

BP use an epistemic framework to model the concept of utility proportional beliefs. The epistemic framework describes the players' beliefs explicitly to obtain a thorough insight into the reasoning underlying the player's decisions.

I introduce the necessary framework to describe static two player normal form games, give a formal notion of utility proportional beliefs, and introduce the concept of common belief in λ-*utility-proportional-beliefs*. Finally, I discuss BP's algorithm to find the beliefs that satisfy the notion of common belief in λ-*utility-proportional-beliefs*.

To make it more convenient for the reader, I will use the notation introduced by BP and state their definitions, theorems and lemmas without modification.

2.1 Epistemic model

BP follow the type-based approach to epistemic game theory. Within the type-based approach a set of types is assigned to every player. Each type corresponds to a belief on the opponents choice-type combinations. Consequently, every type induces an infinite belief hierarchy. Moreover, BP follow the one-player perspective–an approach that has been strongly advocated by Perea (2007b), (2007a), and (2012). Hence, the epistemic concepts are modeled as mental states inside the mind of a single person. This approach seems to be intuitive since the epistemic framework represents a player's beliefs and the reasoning process that takes place in the reasoner's mind preceding any choice. BP's approach can be formalized as follows.

A finite normal form game for two players is represented by a tuple

$$\Gamma = (I, (C_i)_{i \in I}, (U_i)_{i \in I})$$

where I denotes a finite set of players, C_i denotes player i's finite choice set, and $U_i : \times_{j \in I} C_j \to \mathbb{R}$ denotes player i's utility function. As I am only interested in two player games we have $I = \{1, 2\}$.

On top of the normal form game they define an epistemic model, with which various epistemic mental states of players can be described.

Definition 1. *An epistemic model of a game* Γ *is a tuple* $\mathcal{M}^\Gamma = ((T_i)_{i \in I}, (b_i)_{i \in I})$*, where*

- T_i *is a set of types for player* $i \in I$

– b_i: $T_i \to \Delta(C_j \times T_j)$ *assigns to every type* $t_i \in T_i$ *a probability measure with finite support on the set of opponents' choice-type combinations.*

The finite support condition restricts the focus on types that only assign positive probability to finitely many types for their opponent. This restriction avoids the introduction of a σ-algebra and therefore additional topological assumptions. The probability measure $b_i(t_i)$ represents type t_i's belief function on the set of opponents' choice-type pairs. For convenience $b_i(t_i)$ also denotes any projected belief function for type t_i, e.g. $(b_i(t_i))(c_j|t_j)$ gives the probability that player i assigns to j's choice c_j given that j is of type t_j. It will be clear from the context what $b_i(t_i)$ refers to.

Now consider a game Γ, an epistemic model $\mathcal{M}^\Gamma$ of it, and fix two players $i, j \in I$ such that $i \neq j$. A type $t_i \in T_i$ of i is said to *deem possible* some type $t_j \in T_j$ of her opponent j, if $b_i(t_i)$ assigns positive probability to an opponents' choice-type combination that includes t_j. The set $T_j(t_i)$ describes the set of types of player j deemed possible by t_i.

Player i's *conditional belief* that player j chooses c_j conditional on her type being t_j is expressed as

$$(b_i(t_i))(c_j|t_j) := \frac{(b_i(t_i))(c_j, t_j)}{(b_i(t_i))(t_j)}$$

given a type $t_i \in T_i$ of player i, the opponent's type $t_j \in T_j(t_i)$, and her belief that j is of type t_j.

Type t_i's expected utility given her belief on her opponents' choice combinations is given by

$$u_i(c_i, t_i) = \sum_{c_j \in C_j} (b_i(t_i))(c_j) U_i(c_i, c_j).$$

Additionally, type t_i's average utility is denoted by

$$u_i^{average}(t_i) := \frac{1}{|C_i|} \sum_{c_i \in C_i} u_i(c_i, t_i).$$

Finally, $C := \times_{i \in I} C_i$ denotes all the choice combinations in the game. The best and the worst possible utilities of player i are denoted as $\bar{u}_i := \max_{c \in C} u_i(c)$ and $\underline{u}_i := \min_{c \in C} u_i(c)$. This concludes the formal representation of the epistemic model for utility proportional beliefs. We are now ready to introduce the concept itself.

2.2 Common belief in utility proportional beliefs

The core idea of the concept of utility proportional beliefs is that players assign probabilities to their opponents' choices proportional to the utilities these choices yield for their opponents. The idea is formalized using the framework of an epistemic model for a two-player normal form game I introduced before.

Definition 2. *Let* $i, j \in I$ *be the two players, and* $\lambda_j \in \mathbb{R}$ *such that* $\lambda_j \geq 0$. *A type* $t_i \in T_i$ *of player* i *expresses* λ_j*-utility-proportional-beliefs, if*

$$(b_i(t_i))(c_j|t_j) - (b_i(t_i))(c'_j|t_j) = \frac{\lambda_j}{\bar{u}_j - \underline{u}_j}(u_j(c_j, t_j) - u_j(c'_j, t_j)) \qquad (2.1)$$

for all $t_j \in T_j(t_i)$, *for all* c_j, $c'_j \in C_j$.

The definition directly corresponds to the idea of utility proportional beliefs: the difference in probabilities player i assigns to the opponents' choices is equal to the difference of the normalized utilities times the proportionality factor λ_j. This interpretation differs slightly from BP's in the sense that they call $\lambda_j/(\bar{u}_j - \underline{u}_j)$ the proportionality factor instead of λ_j. However, $1/(\bar{u}_j - \underline{u}_j)$ normalizes the utilities such that maximal difference between the highest and lowest utility is equal to one and that beliefs are invariant with respect to affine transformations of the utilities (see BP). BP give an intuitive interpretation of the proportionality factor λ_j as measure of the sensitivity of a player's beliefs to differences in the opponents utilities.

Note that there exists an upper bound for the proportionality factor called λ_j^{max}. It is the maximum value of λ_j for which equation (2.1) yields well-defined probability measures. Setting λ_j larger than λ_j^{max} can lead to probabilities larger than 1 or even negative. The lower limit of λ_j is 0. In this case the differences in utilities do not play a role in the belief formation process. Players simply assign equal probability to their opponents' choices.

BP also present an explicit formula for the belief about a given opponent's choice conditional of her being of a given type. It relates the conditional belief in a specific opponent's choice to the utility that this choice generates for the respective opponent.

Lemma 1. *Let $i, j \in I$ be the two players, and $\lambda_j \in \mathbb{R}$. A type $t_i \in T_i$ of player i expresses λ_j-utility-proportional-beliefs if and only if*

$$(b_i(t_i))(c_j|t_j) = \frac{1}{|C_j|} + \frac{\lambda_j}{\bar{u}_j - \underline{u}_j}(u_j(c_j, t_j) - u_j^{average}(t_j)), \tag{2.2}$$

for all $t_j \in T_j(t_i)$, for all $c_j \in C_j$, and for $j \in I\backslash\{i\}$.

Equation (2.2) gives an intuitive interpretation of conditional beliefs under utility proportional beliefs: player i assigns to her opponent's type the uniform distribution on the respective opponent's choice set plus or minus an adjustment for each choice depending on its goodness relative to the average utility.

The concept of common belief in λ-*utility-proportional-beliefs* requires that both players entertain utility proportional beliefs, that both players believe their opponent holds utility proportional beliefs, that both players believe their opponents believe that their opponents do so, and so on. This idea is formally expressed by Definition 3.

Definition 3. *Let $i, j \in I$ be the two players, $t_i \in T_i$ be some type of player i, and $\lambda = (\lambda_j)_{j \in I} \in \mathbb{R}^{|I|}$.*

- *Type t_i expresses 1-fold belief in* λ-utility-proportional-beliefs*, if t_i expresses λ_j-*utility-proportional-beliefs*.*

- *Type t_i expresses k-fold belief in* λ-utility-proportional-beliefs*, if $(b_i(t_i))$ only deems possible types $t_j \in T_j$ for j such that t_j expresses $k-1$-fold belief in* λ-utility-proportional-beliefs*, for all $k > 1$.*

- *Type t_i expresses common belief in* λ-utility-proportional-beliefs*, if t_i expresses k-fold belief in* λ-utility-proportional-beliefs *for all $k \geq 1$.*

Definition 3 tells us that a type satisfying common belief in λ-*utility-proportional-beliefs* has λ_j-*utility proportional-beliefs*, beliefs that her opponent has λ-*utility-proportional-beliefs*, belief that her opponent beliefs that she holds λ-*utility-proportional-beliefs*, and so on.

Finally, Definition 3 describes all choices player i can rationally make under common belief in λ-*utility-proportional-beliefs*.

Definition 4. *Let $i, j \in I$ be the two players, and $\lambda = (\lambda_i)_{i \in I} \in \times_{i \in I}\mathbb{R}$. A choice $c_i \in C_i$ of player i is rational under common belief in* λ-utility-proportional-beliefs*, if there exists an epistemic model $\mathcal{M}^\Gamma$ and some type $t_i \in T_i$ of player i such that c_i is optimal given $(b_i(t_i))$ and t_i expresses common belief in* λ-utility-proportional-beliefs.

2.3 Algorithm

BP introduce an algorithm to find exactly those beliefs that are possible under common belief in λ-*utility-proportional-beliefs*. The algorithm iteratively deletes beliefs so that only the beliefs, which are possible under common belief in λ-*utility-proportional-beliefs*, survive. To introduce the algorithm some further notation is required.

By $P_i^0 := \Delta(C_j)$ BP denote the set of i's beliefs about her opponent's choice combinations. Given $p_i \in P_i^0$ they define $u_i^{average}(p_i) := \sum_{c_i \in C_i} u_i(c_i, p_i) / |C_i|$. Furthermore, they define for player i and her opponent $j \in I \setminus \{i\}$, $p_i^* : \Delta(C_i) \to \Delta(C_j)$, where p_i^* is a function mapping the beliefs of player j on her opponent's choice combinations to beliefs on j's choice:

$$(p_i^*(p_j))(c_j) := \frac{1}{|C_j|} + \frac{\lambda_j}{\overline{u}_j - \underline{u}_j}(u_j(c_j, p_j) - u_j^{average}(p_j)) \tag{2.3}$$

for all $c_j \in C_j$ and for all $p_j \in P_j^0$. These are all the ingredients needed to state the algorithm for *iterated elimination of* λ-*utility-disproportional-beliefs* formally.

Definition 5. *For both players $i \in I$ and for all $k \geq 0$ the set P_i^k of i's beliefs about her opponent's choice combinations is inductively defined as follows:*

$$\begin{aligned} P_i^0 &:= \Delta(C_j), \text{ and} \\ P_i^k &:= p_i^*(P_j^{k-1}). \end{aligned}$$

The set of beliefs $P_i^\infty = \cap_{k \geq 0} P_i^k$ contains the beliefs that survive iterated elimination of utility-disproportional-beliefs.

By $p_i^*(P_j^{k-1})$ BP denote the set $\{p_i^*(p_j) : p_j \in P_j^{k-1}\}$, where $p_i^*(p_j)$ is the utility-proportional belief on j's choice generated by p_j. Using the algorithm repeatedly deletes beliefs that are not utility proportional with respect to beliefs from the preceding set of beliefs generated by the algorithm. Actually, it iteratively deletes beliefs p_i that cannot be obtained by the function p_i^* on the set P_j^{k-1}.

Theorem 1 in BP establishes that this algorithm yields precisely those beliefs that a player can hold under common belief in λ-*utility-proportional-beliefs*. Furthermore, their Theorem 1 establishes that common belief in λ-*utility-proportional-beliefs* is always possible. Hence, the concept can be applied to describe the players' reasoning in any static game.

3 Insights into the players' reasoning processes

As shown above, one can find a player's belief expressing common belief in λ-*utility-proportional-beliefs* by applying BP's algorithm that iteratively eliminates utility-disproportional beliefs. This iterative procedure, however, allows only for limited insight into the players' reasoning process. From a behavioral point of view, it is, however, interesting to understand the subtle implications of the concept. For instance, *"How do players reason about their opponents?"*

Fortunately, the concept of utility proportional beliefs has a property that helps us to further investigate the players' reasoning process. BP show in their Theorem 2 that beliefs are unique in the two player case. Moreover, I use their Lemma 2 to find the unique beliefs under common belief in λ-*utility-proportional-beliefs* directly. A direct formula would allow us to further investigate the players' reasoning under the concept of common belief in λ-*utility-proportional-beliefs*.

First of all I will, however, discuss why Lemma 2 guarantees the existence of a fixed point. Secondly, I will derive an explicit formula for the unique beliefs in the two player case and thirdly I will examine it to learn more about the players' reasoning process.

3.1 Existence of a unique fixed point

To understand why there exists a unique fixed point we will investigate BP's Lemma 2 more closely. To do so, we first need to establish some notation. Remember, that P_i^k denotes the set of beliefs generated for player i in round k of the algorithm. Furthermore, BP define for any two sets $A, B \subseteq P_i^0$ and for all $\alpha \in [0,1]$ the set $\alpha A + (1-\alpha)B := \{\alpha a + (1-\alpha)b : a \in A \text{ and } b \in B\}$. Finally, $p_i^*(\cdot, \lambda_j)$ denotes the function p_i^* induced by the proportionality factor λ_j.

Lemma 2. *Let Γ be a two player game with $I = \{1,2\}$, $\lambda_1 < \lambda_1^{max}$, and $\lambda_2 < \lambda_2^{max}$. Moreover, define $\alpha_1 := \lambda_1/\lambda_1^{max} < 1$, $\alpha_2 := \lambda_2/\lambda_2^{max} < 1$, and $\alpha := \max\{\alpha_1, \alpha_2\} < 1$. Then, for every player $i \in I$ and every round $k \geq 0$ there exists $p_i \in P_i^0$ such that $P_i^k \subseteq \alpha^k P_i^0 + (1-\alpha^k)\{p_i\}$.*

Lemma 2 states that for the k^{th} iteration of the algorithm there exists some belief $p_i \in P_i^0$ such that the convex combination $\alpha^k P_i^0 + (1-\alpha^k)\{p_i\}$ contains all beliefs that a player can hold under up to *k-fold belief in* λ-*utility-proportional-beliefs*. For the case of common belief in λ-*utility-proportional-beliefs*, we let k go to infinity and obtain

$$\lim_{k\to\infty} \alpha^k P_i^0 + (1-\alpha^k)\{p_i^k\} = \{p_i^\infty\} \tag{3.1}$$

for some $p_i^\infty \in P_i^0$ where p_i^k denotes a belief $p_i^k \in P_i^0$ such that $P_i^k \subseteq \alpha^k P_i^0 + (1-\alpha^k)\{p_i^k\}$ holds. We observe that the convex combination converges to a singleton and hence to a unique fixed point. Now suppose $p_i' \in P_i^\infty$ is the belief of i and $p_j' \in P_j^\infty$ is the belief of j for which equation (3.1) holds. Recall, that by the construction of the algorithm it holds that $P_i^k = p_i^*(P_j^{k-1}, \lambda_j)$. Then it holds that $p_i^*(p_j') = p_i'$ and $p_j^*(p_i') = p_j'$ and hence it also holds that $p_i^*(p_j^*(p_i')) = p_i'$.

Consequently, the mapping $p_i^* \circ p_j^*$ has a unique fixed point, in two-player games where the players' are holding common belief in λ-*utility-proportional-beliefs*.

3.2 Formula for beliefs

I just showed that there exists a unique fixed point for the mapping $p_i^* \circ p_j^*$ under common belief in λ-*utility-proportional-beliefs*. Now I will develop the formula to obtain the beliefs that player i can hold under common belief in λ-*utility-proportional-beliefs* directly.

Our goal is to express the mapping (2.3) in matrix notation so that we can express the beliefs over all choice combinations simultaneously. Before we can proceed we need to introduce some additional notation. I will denote $((p_i^*(p_j))(c_j))_{c_j \in C_j}$ by the vector $p_i^*(p_j)$. For every $n \times m$ game Γ I denote the number of choices of player i by $n = |C_i|$ and the number of choices for player j by $m = |C_j|$. Moreover, let $N = \{1, ..., n\}$ and $M = \{1, ..., m\}$. The $n \times 1$ vector i_n with $i_n = (\frac{1}{n}, ..., \frac{1}{n})$ is the vector equivalent to $1/|C_i|$ in (2.3). Let $C_i = \{c_i^1, \ldots, c_i^n\}$ and $C_j = \{c_j^1, \ldots, c_j^m\}$ so that we can denote player i's

$n \times m$ utility matrix by

$$U_i = \begin{bmatrix} U_i(c_i^1, c_j^1) & \cdots & U_i(c_i^1, c_j^m) \\ \vdots & & \vdots \\ U_i(c_i^n, c_j^1) & \cdots & U_i(c_i^n, c_j^m) \end{bmatrix}. \tag{3.2}$$

Now we can express the expected utility as a $n \times 1$ vector $u_i(p_i) = U_i p_i$, the average utility as a $n \times 1$ vector $u_i^{average} = \frac{1}{n}\mathbb{1}_n u_i(p_i)$, where $\mathbb{1}_n$ represents a $n \times n$ matrix of 1's. After substituting $u_i(p_i)$ into $u_i^{average}$ and rearranging we obtain

$$u_i(p_i) - u_i^{average} = (I_n - \frac{1}{n}\mathbb{1}_n)U_i p_i,$$

where I_n is the $n \times n$ identity matrix. The centering term $(I_n - \frac{1}{n}\mathbb{1}_n)$ will play a crucial role in the players' reasoning. Hence, we will denote the term by the $n \times n$ centering matrix $C_n := (I_n - \frac{1}{n}\mathbb{1}_n)$. The matrix C_n has $\frac{n-1}{n}$ on the diagonal and $-\frac{1}{n}$ off the diagonal. Intuitively, the centering matrix subtracts the mean from the columns of a matrix when left multiplied. Now can write $p_i^*(p_j)$ as

$$p_i^*(p_j) = i_m + \frac{\lambda_j}{\bar{u}_j - \underline{u}_j} C_m U_j p_j. \tag{3.3}$$

Since $p_i^*(p_j)$ maps j's beliefs about i into i's beliefs about j, and $p_j^*(p_i)$ maps i's beliefs about j into j's beliefs about i, we can substitute $p_j^*(p_i)$ for p_j, so that we obtain $p_i^*(p_j^*(p_i))$. We could again substitute p_i with $p_j^*(p_j)$, this, however, would result in an infinite process. Instead we know that there exists a unique fixed point for $p_i^* \circ p_j^*$. Suppose $p_i' \in P_i^\infty$ is the fixed point of $p_i^* \circ p_j^*$ then we can solve $p_i' = p_i^*(p_j^*(p_i'))$ for p_i' and obtain i's beliefs about j under common belief in λ-*utility-proportional-beliefs*. We substitute p_j by player j's belief generating function in (3.3) and obtain

$$\begin{aligned} p_i' = p_i^*(p_j^*(p_i')) &= i_m + \frac{\lambda_j}{\bar{u}_j - \underline{u}_j} C_m U_j p_j^*(p_i') \\ &= i_m + \frac{\lambda_j}{\bar{u}_j - \underline{u}_j} C_m U_j \left[i_n + \frac{\lambda_i}{\bar{u}_i - \underline{u}_i} C_n U_i p_i' \right]. \end{aligned}$$

Solving for p_i' yields

$$p_i' = (I_m - \frac{\lambda_j \lambda_i}{(\bar{u}_j - \underline{u}_j)(\bar{u}_i - \underline{u}_i)} C_m U_j C_n U_i)^{-1}(i_m + \frac{\lambda_j}{\bar{u}_j - \underline{u}_j} C_m U_j i_n). \tag{3.4}$$

Hence, we can express player i's beliefs about player j as an explicit formula.

3.3 Understanding the players' reasoning

Equation (3.4) gives an explicit expression of how players' form their beliefs under common belief in λ-*utility-proportional-beliefs*. Key to developing an understanding about how players form their beliefs is to recall the meaning of λ_j. The proportionality factor λ_j affects the extent to which player i takes j's differences in utility into account when reasoning about player j. To have a standardized measure, I will use α_i as defined in Lemma 2 instead of λ_i because we need $\lambda_i < \lambda_i^{max}$ to hold and λ_i^{max} differs from game to game. Remember λ_i^{max} is defined so that $p_j^*(p_i)$ always yields well-defined probability measures. Consequently, I write $\alpha_i \lambda_i^{max}$ instead of λ_i with $\alpha_i \in [0, 1]$ to have a standardized measure for the players' sensitivity to differences in the utilities of their respective opponents. The fixed-point expression (3.4) then becomes

$$p_i' = (I_m - \frac{(\alpha_j \alpha_i)(\lambda_j^{max} \lambda_i^{max})}{(\bar{u}_j - \underline{u}_j)(\bar{u}_i - \underline{u}_i)} C_m U_j C_n U_i)^{-1} (i_m + \frac{\alpha_j \lambda_j^{max}}{\bar{u}_j - \underline{u}_j} C_m U_j i_n).$$

To save some notation I define $U_i^{norm} = (\lambda_i^{max} / (\overline{u}_i - \underline{u}_i)) U_i$. Now we can define $\beta_i(U_i, U_j)$ as the function for i's belief about j under common λ-*utility-proportional-beliefs* as

$$\beta_i(\alpha_i, \alpha_j, U_i, U_j) := (I_m - \alpha_j \alpha_i C_m U_j^{norm} C_n U_i^{norm})^{-1} (i_m + \alpha_j C_m U_j^{norm} i_n).$$

With $\beta_i(\alpha_i, \alpha_j, U_i, U_j)$ we obtain i's beliefs about j directly given their respective payoff matrices and proportionality factors. This expression, however, still involves the inverse of a matrix which makes the interpretation quite difficult. Hence, we further simplify the notation and show that the inverse can also be expressed as an infinite sum.

Firstly, we define the matrix $G_j := \alpha_j C_m U_j^{norm}$ since this expression is repeated several times in the expression above, it is useful to develop a more intuitive understanding. By left-multiplying the normalized utility matrix U_j^{norm} with the centering matrix C_m, one obtains a matrix where for every element the average of its column has been subtracted. Note that the rows of U_j^{norm} correspond to i's choices and the columns to j's choices. The same holds for the matrix $C_m U_j^{norm}$, only that now each element represents the relative goodness of a choice given an opponent's choice. Therefore, the matrix G_j gives the goodness of a choice given an opponent's choice, scaled by the sensitivity to the opponents differences in utility α_j.

To obtain an expression for the players' beliefs that does not rely on the inverse of a matrix, I show that the inverse can also be expressed as an infinite sum. Thus, we define the $m \times m$ matrix $S_j := G_j G_i$, which allows us to rewrite the inverse as $(I_m - S_j)^{-1}$. To show that $(I_m - S_j)^{-1} = I_m + S_j + S_j^2 + S_j^3 + \cdots$ holds, we first have to convince ourselves that the infinite sum of matrices always converges.

Lemma 3. *It holds that* $(p_i^* \circ p_j^*)^k (p_i) = (i_m + G_j i_n) + S_j (i_m + G_j i_n) + \cdots + S_j^{k-1} (i_m + G_j i_n) + S_j^k p_i$ *for all* $p_i \in P_i^0$ *for all* $k = \{1, \ldots, \infty\}$.

Proof. See Appendix. □

Lemma 4. *The infinite sum* $(I_m + \sum_{n=1}^{\infty} S_j^n)$ *always converges to* $(I_m - S_j)^{-1}$.

Proof. See Appendix. □

Now we can state the direct belief using the infinite sum instead of the inverse

$$\begin{aligned} \beta_i &= \sum_{k=0}^{\infty} (G_j G_i)^k (i_m + G_j i_n) \\ &= (i_m + G_j i_n) + G_j G_i (i_m + G_j i_n) \\ &\quad + G_j G_i \left[G_j G_i (i_m + G_j i_n) \right] + \cdots . \end{aligned} \tag{3.5}$$

We see that the expression $(i_m + G_j i_n)$ is repeated several times. Note that $(i_m + G_j i_n)$ is (3.3) applied to the uniform belief about player i's choices. In the second term, this expression is then adjusted by left multiplying the matrices $G_j G_i$. In the third term the second term is adjusted by left multiplying $G_j G_i$, and so on. Therefore, we call $(i_m + G_j i_n)$ the initial belief, $\beta_i^{initial}$. To further emphasize this process, we define β_i^k as the belief that player i holds after the kth reasoning step,

$$\begin{aligned} \beta_i^0 &:= \beta_i^{initial} \\ \beta_i^k &:= \beta_i^{initial} + G_j G_i (\beta_i^{k-1}), \end{aligned} \tag{3.6}$$

such that $\lim_{k \to \infty} \beta_i^k = \beta_i$ holds. To provide a clear insight, I inspect the initial belief and the process leading to the final belief individually.

3.3.1 The initial belief

We call $\beta_i^{initial}(\alpha_j, U_j) = i_m + \alpha_j C_m U_j^{norm} i_n$ the initial belief because $\beta_i^{initial}$ is the core of the step-wise adjustment process described in (3.6). Moreover, it shows how player i constructs her beliefs about player j without taking into account that player j reasons about her. This is the same as when player i would believe that player j is insensitive to her differences in utilities in the sense that $\alpha_i = 0$, so that $\beta_i = \beta_i^{initial}$. In this case, player i's belief about player j only depends on j's payoff matrix and not on her own. Consequently, player i does not consider any strategic interaction when reasoning about j.

Let us first consider the simplest case when player i is insensitive to differences in j's utilities, i.e. $\alpha_j = 0$. In this case, we have $\beta_i^{initial} = i_n$. Recall that the $n \times 1$ vector $i_n = (\frac{1}{n}, ..., \frac{1}{n})$ assigns a uniform distribution over j's choice-combinations. Hence, if $\alpha_j = 0$ player i deems all choices of player j equally likely.

Now let us consider the case when $\alpha_j > 0$. In this scenario player i takes j's payoff scheme into account when forming her beliefs. As in the previous case player i starts off by assigning equal probability to her opponent's choice combinations. In this case, however, the uniform assignment will be corrected by adding the term $G_j i_n$. As discussed above, G_j gives the relative goodness of j's choices given i's choice. We obtain the goodness of j's choices given that player j assigns equal probability i's choices. Therefore, when player i forms her belief about player j she assumes that j deems each of her choices equally likely when assessing the goodness of j's choices given i's. By adding up the vectors $G_j i_n$ and i_m we obtain the beliefs that player i holds about j when we only consider the initial belief. Intuitively, player i first assigns equal probability to all of j's choices and then corrects these probabilities according to the average goodness of j's choices.

3.3.2 The belief formation process

Key to an understanding of the reasoning process under common belief in utility proportional beliefs is to understand the process defined in (3.6). The process starts with $\beta_i^{initial}$, which is the basis for further reasoning and therefore not considered a reasoning step itself. In the first reasoning step player i adjusts the initial belief by adding the initial belief left multiplied by $G_j G_i$. To obtain a more intuitive understanding we rewrite β_i^1 as follows

$$\beta_i^1 = i_m + G_j(i_n + G_i(\beta_i^0)).$$

We see that first player i takes j's perspective, which is reflected in the expression $i_n + G_i(\beta_i^0)$. Here player j forms a belief about player i given i's belief about j from the previous reasoning step, the initial belief. First j assigns equal probability to all of i's choices. Then she corrects these beliefs by the goodness of i's choices given i's belief about j from the previous reasoning step. The result is a new belief of j about i. Then player i takes her own perspective and assigns equal probability to all of j's choices. These probabilities are then again corrected by the goodness of j's choices given the new belief of j about i.

In the second reasoning step, we have a very similar procedure. The second reasoning step can best be expressed as

$$\beta_i^2 = i_m + G_j(i_n + G_i(\beta_i^1)).$$

As one can immediately see, the second step follows exactly the same process as the first. The basis for the reasoning step is i's belief about j from the previous reasoning step. Based on this belief, i takes j's perspective and forms a belief about herself. This belief is then used to form a new belief about j, resulting in the final belief that i holds about

j after the second reasoning step. This exact procedure is then being repeated for every single reasoning step.

It is also important to note that later reasoning steps will be less important for the final belief than earlier ones. Note that $G_j = \alpha_i \lambda_i^{max} C_m U_j^{norm}$, so that (3.5) can be written as

$$\beta_i = \sum_{k=0}^{\infty} (\alpha_i \alpha_j \lambda_i^{max} \lambda_j^{max} C_m U_j^{norm} C_n U_i^{norm})^k \beta_i^{initial}.$$

Since $\alpha_i, \alpha_j \in [0,1)$, later terms in $\sum_{k=0}^{\infty} (\alpha_i \alpha_j \lambda_i^{max} \lambda_j^{max} C_m U_j C_n U_i)^k$ will be smaller than earlier ones and therefore less important for the final belief β_i. This has also an important implication for the meaning of the proportionality factor α_i: the lower the value of α_i the fewer steps of reasoning a player will undergo to approximate the final belief within a reasonable bound. The same holds true for her opponent's proportionality factor.

4 Examples

In this section I want to analyze and describe the players' actual reasoning process under λ-*utility-proportional-beliefs*. Therefore, we look at games from Goeree and Holt (2001), hereafter GH, which are also analyzed in BP. GH examine one-shot games where the predictions of Nash equilibrium and its refinements give inaccurate predictions of the experimental outcomes. BP conjecture that the concept of utility proportional beliefs yields better predictions of human behavior in experiments than the classical concept of Nash Equilibrium. The goal of this section is to understand how players form their beliefs under common belief in λ-*utility-proportional-beliefs* so that the predicted outcome describes actual outcomes more accurately.

Example 1. First of all, we will look at the asymmetric matching pennies game in Figure 4.1. The unique Nash equilibrium predicts that the *Row Player* believes that the *Column Player* will play *left* with a probability of 1/8 and *right* with a probability of 7/8. The *Column Player* believes that the *Row Player* will play *top* and *bottom* with equal probability. GH, however, observe in their experiments that approximately 95 % of the *Row Players* choose top and 85 % of the *Column Players* choose *right*. Intuitively, this outcome seems reasonable because the *Row Player* is likely to have a higher expected payoff from playing *top* than from playing *bottom*. The *Column Player* can easily anticipate this reasoning and react by playing *right*.

BP's concept of utility proportional beliefs yields a prediction similar to the experimental findings. It predicts that the *Row Player* believes that the *Column Player* chooses *left* with a probability of 0.37 and *right* with a probability of 0.63. For the *Column Player* it predicts that she believes that the *Row Player* will choose *top* with a probability of 0.63 and *bottom* with a probability of 0.37. The resulting expected payoffs suggest that the *Row Player* chooses *top* and that the *Column Player* chooses *right*. These predictions are a lot closer to the outcome of GH's experiment.

		Column Player	
		left	*right*
Row Player	*top*	320, 40	40, 80
	bottom	40, 80	80, 40

Figure 4.1: Asymmetric matching pennies

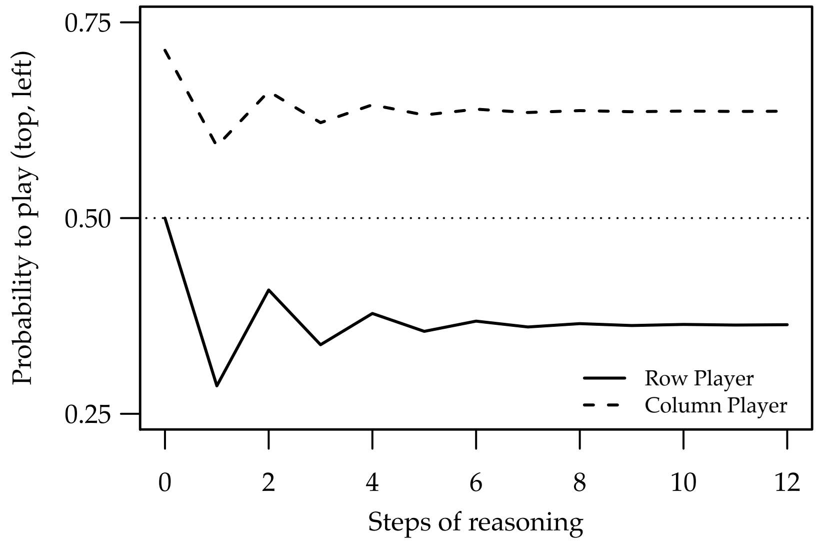

Figure 4.2: Players belief formation process

To understand how the two players form their belief under common belief in λ-*utility-proportional-beliefs*, we first construct their goodness of choice matrices G_R for the *Row Player* and G_C for the *Column Player*. To obtain a good understanding for the reasoning under common belief in λ-*utility-proportional-beliefs*, we want the players to be as sensitive as possible to the differences in their opponent's payoffs. Hence, we set $\alpha_R = \alpha_C = 1 - \epsilon$ where $\epsilon \in \mathbb{R}$ is an infinitesimal small number such that we have $\alpha_R < 1$ and $\alpha_C < 1$. For the *Row Player* we have

$$G_R = \alpha_R \frac{\lambda_R^{max}}{\overline{u}_R - \underline{u}_R} C_2 U_R = \begin{bmatrix} 0.5 & -0.07 \\ -0.5 & 0.07 \end{bmatrix}$$

and for the *Column Player*

$$G_C = \alpha_C \frac{\lambda_R^{max}}{\overline{u}_C - \underline{u}_C} C_2 U_C = \begin{bmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}.$$

For convenience all presented numbers are rounded. Let us now examine how the *Row Player* forms her beliefs about the *Column Player*.

Row Player: The initial belief assigns equal probability to both of the *Column Player*'s choices. The *Row Player* beliefs initially that the *Column Player* believes that she chooses *top* and *bottom* with equal probability. But when the *Column Player* deems both choices equally likely her expected payoff will be 0 for both of her choices. As the goodness of the *Columns Player*'s choices alone does not yield any additional information, the *Row Player* deems both of her choices equally likely in her initial belief. Note that it is just a coincidence that the *Row Player*'s initial belief deems both of the *Column Player*'s choices equally likely, which is due to the fact that the *Column Player* is indifferent between her choices when she beliefs that the *Row Player* play both choices with equal probability.

Figure 4.2 shows how the *Row Player*'s belief develops with each reasoning step. After the seventh reasoning step the belief stabilizes and changes only marginally in the following reasoning steps. The unique belief of the *Row Player* under common belief in λ-*utility-proportional-beliefs* is $\begin{bmatrix} 0.37 & 0.63 \end{bmatrix}$. Consequently, the *Row Player* deems the *Column Player*'s choice *right* more likely than *left*.

Column Player: Again, we first consider the initial belief of the *Column Player* about the *Row Player*. As the *Row Player*'s payoffs are quite distinct the *Column Player* deems the *Row Player*'s choice *top* significantly more likely than her choice *bottom*, we have $\beta_C^{initial} = \begin{bmatrix}0.71 & 0.29\end{bmatrix}^t$. This outcome is quite intuitive when the *Column Player* beliefs that the *Row Player* does not consider any strategic interaction and deems the *Column Player*'s choices equally likely.
Also the *Column Player*'s beliefs converge quickly (see Figure 4.2). Interestingly, the *Column Player*'s beliefs do not fluctuate as strongly as the *Row Player*'s and converge quicker. The resulting unique belief under common belief in λ-*utility-proportional-beliefs* is $\begin{bmatrix}0.63 & 0.37\end{bmatrix}$. Hence, the *Column Player* deems the *Row Player*'s choice *top* more likely than her choice *bottom*.

In my last example I want to discuss a game called *Traveler's Dilemma* which is due to Basu (1994). I introduce the game as in BP. The idea behind the game is that two persons traveled with an identical item. When they arrive back home both discover that their item is broken and ask the airline, which handled their luggage without due care, for compensation. To determine the items' value the airline asks both travelers to submit an integer price between 1 and 10 for the item. The traveler with the lower price receives the named amount and a reward of 2 for being honest. The traveler with the higher price receives an amount equal to the lower price minus a fine of 2. If both submit the same price they both receive an amount equal to the submitted price. Nash equilibrium predicts that the players name a price equal to the lowest price possible. In experiments GH find, however, that with low fines/rewards players name a high price and with high fines/rewards players name a low price.

If we define the fine/reward to be $r \in [0, 4]$, then we can derive the formula for the beliefs with respect to r. We see that the probability assigned to the prices 1 to 4 increases in r. On the other hand, the prices from 6 to 10 decrease in r. The probability assigned to a price of 5 increases until $r = 3$ and decreases afterwards. Therefore, utility proportional beliefs corresponds nicely to the experimental findings.

BP already showed that for a fine/reward of 2 the players will name a price of 6 under common belief in λ-*utility-proportional-beliefs*. I want to show here that for a fine/reward of 4 the players will name a price of 1. Furthermore, I will show that players under common belief in λ-*utility-proportional-beliefs* use 3 steps of reasoning. Since the payoffs for both players are equal, we only need to analyze the reasoning of one player. We call the player i and her opponent j. The goodness of choice matrix G_j is equal to

$$\begin{bmatrix}
0.09 & 0.14 & 0.10 & 0.07 & 0.03 & 0.00 & -0.03 & -0.05 & -0.08 & -0.10 \\
-0.01 & 0.07 & 0.13 & 0.09 & 0.06 & 0.03 & -0.00 & -0.03 & -0.05 & -0.08 \\
-0.01 & -0.03 & 0.06 & 0.12 & 0.08 & 0.05 & 0.02 & -0.00 & -0.03 & -0.05 \\
-0.01 & -0.03 & -0.04 & 0.04 & 0.11 & 0.08 & 0.05 & 0.02 & -0.00 & -0.03 \\
-0.01 & -0.03 & -0.04 & -0.05 & 0.03 & 0.10 & 0.07 & 0.04 & 0.02 & -0.00 \\
-0.01 & -0.03 & -0.04 & -0.05 & -0.06 & 0.03 & 0.10 & 0.07 & 0.04 & 0.02 \\
-0.01 & -0.03 & -0.04 & -0.05 & -0.06 & -0.07 & 0.02 & 0.09 & 0.07 & 0.05 \\
-0.01 & -0.03 & -0.04 & -0.05 & -0.06 & -0.07 & -0.08 & 0.02 & 0.09 & 0.07 \\
-0.01 & -0.03 & -0.04 & -0.05 & -0.06 & -0.07 & -0.08 & -0.08 & 0.02 & 0.10 \\
-0.01 & -0.03 & -0.04 & -0.05 & -0.06 & -0.07 & -0.08 & -0.08 & -0.08 & 0.02
\end{bmatrix}.$$

From the G_j matrix we can confirm the intuition that when one's opponent names a low price, one should also name a low price. Hence, the higher the price the opponent names the higher the price one should choose. The matrix also reflects the fact that one should optimally name a price that is one below the price the opponent named. The resulting initial belief is

$$\beta_i^{initial} = \begin{bmatrix}0.12 & 0.12 & 0.12 & 0.12 & 0.11 & 0.11 & 0.10 & 0.08 & 0.07 & 0.05\end{bmatrix}^t.$$

It shows that player i deems j's choices 1, 2, 3, and 4 more likely than her other choices. When calculating player i's expected payoff given her initial belief about j, we can already see that it would be optimal of her to choose 1.

	β_i^0	β_i^1	β_i^2	β_i
Price 1	0.12	0.14	0.14	0.14
Price 2	0.12	0.13	0.13	0.13
Price 3	0.12	0.13	0.13	0.13
Price 4	0.12	0.12	0.12	0.12
Price 5	0.11	0.10	0.11	0.11
Price 6	0.11	0.10	0.10	0.10
Price 7	0.10	0.09	0.09	0.09
Price 8	0.08	0.07	0.07	0.07
Price 9	0.07	0.07	0.07	0.06
Price 10	0.05	0.05	0.05	0.05

Table 2: Steps of reasoning of player i in the *Traveler's Dilemma* game

With this example I also want to show how player i's belief develops with every step of reasoning. Therefore, we analyze the formation of her beliefs according to equation (3.6). Table 2 shows the beliefs under the first, the second, and the third step of reasoning under common belief in λ-*utility-proportional-beliefs*. After the second step of reasoning, changes in player i's belief about player j are negligibly small. This behavior corresponds to what Nagel (1995) found. Nagel observed in her experiment that people do about 2 to 3 levels of reasoning. It is quite surprising that the simple idea of utility proportional beliefs resembles the experimental findings so well.

The final belief of player i about player j is

$$\beta_i = \begin{bmatrix}0.14 & 0.13 & 0.13 & 0.13 & 0.11 & 0.10 & 0.09 & 0.07 & 0.06 & 0.05\end{bmatrix}^t.$$

When calculating the expected payoff one can infer that player i's choice under common belief in λ-*utility-proportional-beliefs* will be 1.

5 Conclusion

So far most research in game theory has been focused on the final choices of players. The implied reasoning process of a solution concept might, however, be a valuable tool to better understand a solution concept and its features. The two common concepts to model bounded rationality, QRE and CHM, mainly focus on predicting the empirical frequencies of experimental data.

BP's concept, utility proportional beliefs, focuses on the belief formation process itself. The concept is based on a sound idea: the probability assigned to a choice should be proportional to the utilities generated by that choice. Based on this simple idea, one can obtain a formula to calculate the beliefs that players can hold under utility proportional beliefs directly. This formula is consistent with intuitive ideas about reasoning and findings from psychology.

The analysis of the reasoning process brought to light three intuitive and testable assertions about the players' reasoning process. Verifying these assertions will help to improve our understanding of game theoretic solution concepts modeling bounded rationality.

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Appendix

Lemma 3. *It holds that* $(p_i^* \circ p_j^*)^k(p_i) = (i_m + G_j i_n) + S_j(i_m + G_j i_n) + \cdots + S_j^{k-1}(i_m + G_j i_n) + S_j^k p_i$ *for all* $p_i \in P_i^0$ *for all* $k \in \mathbb{N}$.

Proof. We prove Lemma 3 by induction on k. Algebraically, the composed function $p_i^* \circ p_j^*$ corresponds to

$$\begin{aligned} p_i^*(p_j^*(p_i)) &= i_m + G_j(i_n + G_i p_i) \\ &= (i_m + G_j i_n) + S_j p_i. \end{aligned}$$

Hence, we have for $k = 1$ that $(p_i^* \circ p_j^*)(p_i) = (i_m + G_j i_n) + S_j p_i$.

For some $k > 1$, suppose the $k-1$ previous steps were iteratively constructed by substituting $p_i^* \circ p_j^*$ for p_i at every step. Then we have

$$(p_i^* \circ p_j^*)^{k-1}(p_i) = (i_m + G_j i_n) + S_j(i_m + G_j i_n) + S_j^2(i_m + G_j i_n) + \cdots + S_j^{k-1} p_i.$$

Now substitute p_i by $(p_i^* \circ p_j^*)(p_i)$ to obtain step k

$$\begin{aligned} (p_i^* \circ p_j^*)^k(p_i) &= (i_m + G_j i_n) + S_j(i_m + G_j i_n) + \cdots + S_j^{k-1}\left((i_m + G_j i_n) + S_j p_i\right) \\ &= (i_m + G_j i_n) + S_j(i_m + G_j i_n) + \cdots + S_j^{k-1}(i_m + G_j i_n) + S_j^k p_i, \end{aligned}$$

which is what we wanted to prove. □

Lemma 5. *The infinite sum* $(I_m + \sum_{n=1}^{\infty} S_j^n)$ *always converges to* $(I_m - S_j)^{-1}$.

Proof. According to Lemma 3 we have that

$$(p_i^* \circ p_j^*)^k(p_i) = (i_m + G_j i_n) + S_j(i_m + G_j i_n) + \cdots + S_j^{k-1}(i_m + G_j i_n) + S_j^k p_i,$$

for every $k \in \mathbb{N}$ and some $p_i \in P_i^0$. Factoring out $(i_m + G_j i_n)$ yields

$$(p_i^* \circ p_j^*)^k(p_i) = (I_m + S_j + \cdots + S_j^{k-1})(i_m + G_j i_n) + S_j^k p_i.$$

Since BP showed in their Theorem 1 that the iterative application of p_i^* and p_j^* yields exactly those beliefs that the players can hold under common belief in λ-*utility-proportional-beliefs*, and in their Theorem 2 that these beliefs must be unique in the two player case, $(p_i^* \circ p_j^*)^k(p_i)$ must also converge to the unique belief for $k \to \infty$. We then have $\lim_{k\to\infty}(p_i^* \circ p_j^*)^k(p_i) = (I_m + \sum_{n=1}^{\infty} S_j^n)(i_m + G_j i_n)$, which shows that the infinite sum $(I_m + \sum_{n=1}^{\infty} S_j^n)$ exists. It also follows that S_j^k converges to the zero matrix for $k \to \infty$ since $(p_i^* \circ p_j^*)^k(p_i)$ converges to the unique belief for $k \to \infty$ and since S_j^k is the only element affected by k. As the infinite sum $(I_m + \sum_{n=1}^{\infty} S_j^n)$ exists and S_j^k converges to the zero matrix, it follows that the infinite sum $(I_m + \sum_{n=1}^{\infty} S_j^n)$ always exists and that it converges to $(I_m + S_j)^{-1}$. □