ECON915 Microeconomic Theory
Part A: Introduction to Decision Theory
Lecture 2: Utility

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The Idea of Utility Representation

- It is of interest to have a numerical representation of preferences.

- A function $U : X \to \mathbb{R}$ is said to represent the preference relation $\succ \subseteq X \times X$, whenever

$$x \succ y$$

if and only if

$$U(x) > U(y)$$

holds for all $x, y \in X$.

- A function U representing preferences $\succ$ is called a utility function, and $\succ$ is said to have a utility representation.

Decision Theory without Utility?

- It is possible to avoid the notion of utility and to construct a **theory of decisions** based on preferences only.
- Yet, typically utility functions are used rather than preferences to describe an agents's attitude towards alternatives.
- In fact, it is often perceived as more convenient to maximize a numerical function to find the best alternatives for an agent.

Existence of a Utility Representation

- If any preference relation could be represented by a utility function, then utility functions could be used rather than preference relations with no loss of generality.
- **Utility Theory** investigates the possibility of using a numerical function to represent a preference relation and the possibility of numerical representations carrying additional meaning.
- For instance, x is preferred to y more than a is preferred to b.
- The basic question of **utility theory**: Under what assumptions do utility representations exist?

Agenda

- Multiplicity of Utility Functions
- Existence with Finite Sets of Alternatives

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- **Multiplicity of Utility Functions**
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Preference Relations and Utility Functions

- When defining a preference relation from a given utility function, the function has an intuitive meaning that carries with it additional information.
- In contrast, when a utility function is formed to represent a given preference relation, the function has no meaning other than that of representing a preference relation.
- In the latter case, absolute numbers are meaningless, only the relative order is meaningful.
- Indeed, if a preferene relation has a utility representation, then it has an infinite number of such representations.

Multiplicity of Utility Representations

Alternatively phrased, an utility function is ony unique up to a strictly increasing transformation.

Proposition 4

Let $U : X \to \mathbb{R}$ be a utility function of some strict preference relation $\succ \subseteq X \times X$, and $f : \mathbb{R} \to \mathbb{R}$ be some strictly increasing function. Then, the function

$$V : X \to \mathbb{R}$$

such that

$$V(x) := f\big(U(x)\big)$$

for all $x \in X$ also represents $\succ$.

Proof

Observe that for all $x, y \in X$ it is the case that

$$x \succ y,$$

if and only if,

$$U(x) > U(y) \qquad \text{(since } U \text{ represents } \succ\text{)},$$

if and only if,

$$f\big(U(x)\big) > f\big(U(y)\big) \qquad \text{(since } f \text{ is strictly increasing)},$$

if and only if,

$$V(x) > V(y) \qquad \text{(by definition of } V\text{)}.$$

Agenda

- Multiplicity of Utility Functions
- **Existence with Finite Sets of Alternatives**

Representation Theorem for Finite Sets

Proposition 5

Let X be a finite set. A binary relation $\succ \subseteq X \times X$ is a strict preference relation, if and only if, there exists a function $U : X \to \mathbb{R}$ such that for all $x, y \in X$ it is the case that

$$x \succ y,$$

if and only if,

$$U(x) > U(y).$$

Restrictions of Strict Preference Relations are Strict Preference Relations

Lemma 2

Let X be a finite set, $\succ \subseteq X \times X$ be some strict preference relation on X, $Y \subseteq X$ be some subset of X, and $\succ^Y$ be the binary relation $\succ$ restricted to $Y \times Y$. Then, $\succ^Y$ is a strict preference relation on Y.

Proof:

- Consider $a, b \in Y$ such that $a \succ^Y b$. Then, $a \succ b$ by definition of $\succ^Y$. By asymmetry of $\succ$ it follows that $b \not\succ a$. Again, by definition of $\succ^Y$, it obtains that $b \not\succ^Y a$. Thus, $\succ^Y$ is asymmetric.
- Consider $a, b, c \in Y$ such that $a \not\succ^Y b$ and $b \not\succ^Y c$. Then, $a \not\succ b$ and $b \not\succ c$ by definition of $\succ^Y$. By negative transitivity of $\succ$ it follows that $a \not\succ c$. Again, by definition of $\succ^Y$, it obtains that $a \not\succ^Y c$. Thus, $\succ^Y$ is negative transitive.

Restrictions of Weak Preference Relations are Weak Preference Relations

Lemma 3

Let X be a finite set, $\succsim \subseteq X \times X$ be some weak preference relation on X, $Y \subseteq X$ be some subset of X, and $\succsim^Y$ be the binary relation $\succsim$ restricted to $Y \times Y$. Then, $\succsim^Y$ is a weak preference relation on Y.

Proof:

- Consider $a, b \in Y$. Then, $a \succsim b$ or $b \succsim a$ by completeness of $\succsim$. By definition of $\succsim^Y$, it directly follows that $a \succsim^Y b$ or $b \succsim^Y a$, hence $\succ^Y$ is complete.

- Consider $a, b, c \in Y$ such that $a \succsim^Y b$ and $b \succsim^Y c$. Then, $a \succsim b$ and $b \succsim c$ by definition of $\succsim^Y$. By transitivity of $\succsim$ it follows that $a \succsim c$. Again, by definition of $\succsim^Y$, it obtains that $a \succsim^Y c$. Thus, $\succsim^Y$ is transitive.

Existence of Minimal Elements

Lemma 4

Let X be a finite set, and $\succsim \subseteq X \times X$ be some weak preference relation on X. Then, there exists a minimal element, i.e. $a \in X$ such that $x \succsim a$ for all $x \in X$.

Proof of Lemma 4

Induction Basis:

- If $\mid X \mid= 1$, then $x \in X$ is a minimal element, as $x \succsim x$ holds by completeness.

Induction Step:

- Let $\mid X \mid= n$ and consider some $x \in X$.
- By Lemma 3, $\succsim$ restricted to the set $X \setminus \{x\}$ is a weak preference relation and, by the induction hypothesis, $X \setminus \{x\}$ has a minimal element a.
- If $x \succsim a$, then a is minimal in X, too.
- If $x \not\succsim a$, then $a \succsim x$ by completeness, and by transitivity x is minimal in X.

Existence of Maximal Elements

Lemma 5

Let X be a finite set, and $\succsim \subseteq X \times X$ be some weak preference relation on X. Then, there exists a maximal element, i.e. $b \in X$ such that $b \succsim x$ for all $x \in X$.

Proof of Lemma 5

Induction Basis:

- If $\mid X \mid = 1$, then $x \in X$ is a maximal element, as $x \succsim x$ holds by completeness.

Induction Step:

- Let $\mid X \mid = n$ and consider some $x \in X$.
- By Lemma 3, $\succsim$ restricted to the set $X \setminus \{x\}$ is a weak preference relation and, by the induction hypothesis, $X \setminus \{x\}$ has a maximal element b.
- If $b \succsim x$, then b is maximal in X, too.
- If $b \not\succsim x$, then $x \succsim b$ by completeness, and by transitivity x is maximal in X.

Representation Theorem for Finite Sets

Proposition 5

Let X be a finite set. A binary relation $\succ \subseteq X \times X$ is a strict preference relation, if and only if, there exists a function $U : X \to \mathbb{R}$ such that for all $x, y \in X$ it is the case that

$$x \succ y,$$

if and only if,

$$U(x) > U(y).$$

Proof of the *if* Direction ($\Leftarrow$) of Proposition 5

- Let $U : X \to \mathbb{R}$ be a function such that for all $x, y \in X$ it is the case that $x \succ y$, if and only if, $U(x) > U(y)$.

- Consider $a, b \in X$ such that $a \succ b$. Then, $U(a) > U(b)$, which directly implies that $U(b) \not> U(a)$. It follows that $b \not\succ a$. Hence, $\succ$ is asymmetric.

- Consider $a, b, c \in X$ such that $a \not\succ b$ and $b \not\succ c$. Then, $U(a) \not> U(b)$ and $U(b) \not> U(c)$. It follows that $U(a) \leq U(b)$ and $U(b) \leq U(c)$. Thus, $U(a) \leq U(c)$, and therefore $U(a) \not> U(c)$. Consequently, $a \not\succ c$. Hence, $\succ$ is negative transitive.

Proof of the *only if* Direction ($\Rightarrow$) of Proposition 5

- By induction, it is shown that if $\mid X \mid = n$ and $\succ$ is a preference relation on X, then there exists a function $U : X \to (0;1)$ such that for all $x, y \in X$ it is the case that $x \succ y$, if and only if, $U(x) > U(y)$.

Induction Basis:

- Let $\mid X \mid = 1$ and define $U(x) = \frac{1}{2}$ for $x \in X$.
- Because $\succ$ is asymmetric, $x \not\succ x$ holds.
- It is also the case, that $U(x) \not> U(x)$.
- Therefore, $x \succ y$, if and only if, $U(x) > U(y)$ holds trivially.

Proof of the *only if* Direction ($\Rightarrow$) of Proposition 5

Induction Step:

- Suppose that, if $\mid X \mid = n-1$ and $\succ$ is a preference relation on X, then there exists a function $U : X \to (0;1)$ such that for all $x, y \in X$ it is the case that $x \succ y$, if and only if, $U(x) > U(y)$.

- Let $\mid X \mid = n$ and $\succ$ be a preference relation on X.

- Consider some $x^\circ \in X$ and form the set $X' = X \setminus \{x^\circ\}$, where $\mid X' \mid = n-1$.

- If follows, by Lemma 2, that $\succ$ restricted to $X \setminus \{x^\circ\}$ is a preference relation on $X \setminus \{x^\circ\}$.

- By the induction hypothesis, there exists a function $U' : X' \to (0;1)$ such that for all $x, y \in X'$ it is the case that $x \succ y$, if and only if, $U'(x) > U'(y)$.

Proof of the *only if* Direction ($\Rightarrow$) of Proposition 5

Four exhaustive cases are now considered:

1. There exists $x' \in X'$ such that $x^\circ \sim x'$.
2. $x^\circ \succ x'$ for all $x' \in X'$.
3. $x' \succ x^\circ$ for all $x' \in X'$.
4. $x^\circ \nsim x'$ for all $x' \in X'$ and there exist $x'', x''' \in X'$ such that $x'' \succ x^\circ$ and $x^\circ \succ x'''$.

Proof of the *only if* Direction ($\Rightarrow$) of Proposition 5

Case 1:

- There exists $x' \in X'$ such that $x^\circ \sim x'$.

- Define $U : X \to (0;1)$ such that

$$U(x) = \begin{cases} U'(x), & \text{if } x \in X', \\ U'(x'), & \text{if } x = x^\circ. \end{cases}$$

- Firstly, if $x, y \in X'$, then $x \succ y$, if and only if, $U'(x) > U'(y)$, by the induction hypothesis, if and only if, $U(x) > U(y)$, as U coincides with U' on X'.

- Secondly, if $x \in X'$ and $y = x^\circ$, then $x \succ x^\circ$, if and only if, $x \succ x'$, as $x^\circ \sim x'$, if and only if, $U'(x) > U'(x')$, by the induction hypothesis, if and only if, $U(x) > U(x^\circ)$, by definition of U.

Proof of the *only if* Direction ($\Rightarrow$) of Proposition 5

Case 1 (continued):

- Thirdly, if $x = x^\circ$ and $y \in X'$, then $x^\circ \succ y$, if and only if, $x' \succ y$, as $x' \sim x^\circ$, if and only if, $U'(x') > U'(y)$, by the induction hypothesis, if and only if, $U(x^\circ) > U(y)$, by definition of U.
- Fourthly, if $x = y = x^\circ$, then both $x^\circ \succ x^\circ$ as well as $U(x^\circ) > U(x^\circ)$ are impossible, hence $x \succ y$, if and only if, $U(x) > U(y)$ holds trivially.

Proof of the *only if* Direction ($\Rightarrow$) of Proposition 5

Case 2:

- $x^\circ \succ x'$ for all $x' \in X'$.
- Define $U : X \to (0;1)$ such that

$$U(x) = \begin{cases} U'(x), & \text{if } x \in X', \\ \frac{\max_{x \in X'} \left(U'(x)+1\right)}{2}, & \text{if } x = x^\circ. \end{cases}$$

- Firstly, if $x, y \in X'$, then proceed just as in **Case 1**.
- Secondly, if $x \in X'$ and $y = x^\circ$, then $x \succ x^\circ$ is impossible as $x^\circ \succ x'$ for all $x' \in X'$ by assumption, and $U(x) > U(x^\circ)$ is also impossible by the construction of U, hence $x \succ y$, if and only if, $U(x) > U(y)$ holds trivially.

Proof of the *only if* Direction ($\Rightarrow$) of Proposition 5

Case 2 (continued):

- Thirdly, if $x = x^\circ$ and $y \in X'$, then $x^\circ \succ y$ as $x^\circ \succ x'$ for all $x' \in X'$ by assumption, and $U(x^\circ) > U(y)$, by construction of U.
- Fourthly, if $x = y = x^\circ$, then proceed just as in **Case 1**.

Proof of the *only if* Direction ($\Rightarrow$) of Proposition 5

Case 3:

- $x' \succ x^\circ$ for all $x' \in X'$.
- Define $U : X \to (0;1)$ such that

$$U(x) = \begin{cases} U'(x), & \text{if } x \in X', \\ \frac{\min_{x \in X'} \left(U'(x)\right)}{2}, & \text{if } x = x^\circ. \end{cases}$$

- Proceed analogously to **Case 2**.

Proof of the *only if* Direction ($\Rightarrow$) of Proposition 5

Case 4:

- $x^\circ \not\sim x'$ for all $x' \in X'$ and there exist $x'', x''' \in X'$ such that $x'' \succ x^\circ$ and $x^\circ \succ x'''$.

- By Lemma 4, there exist $\bar{x} \in X'$ such that $\bar{x}$ is $\succsim$-minimal in the set $\{x' \in X' : x' \succ x^\circ\}$, and thus $U'(\bar{x}) = \min_{y \in X' : y \succ x^\circ} \left(U'(y)\right)$

- Also, by Lemma 5, there exist $\underline{x} \in X'$ such that $\underline{x}$ is $\succsim$-maximal in the set $\{x' \in X' : x^\circ \succ x'\}$, and thus $U'(\underline{x}) = \max_{y \in X' : x^\circ \succ y} \left(U'(y)\right)$.

- Define $U : X \to (0;1)$ such that

$$U(x) = \begin{cases} U'(x), & \text{if } x \in X', \\ \frac{U'(\bar{x}) + U'(\underline{x})}{2}, & \text{if } x = x^\circ. \end{cases}$$

Proof of the *only if* Direction ($\Rightarrow$) of Proposition 5

Case 4 (continued):

- Since $\bar{x} \succ x^\circ \succ \underline{x}$, it follows by transitivity of $\succ$, that $\bar{x} \succ \underline{x}$, and thus $U'(\bar{x}) > U'(\underline{x})$ by construction of U.
- Therefore, $U(\bar{x}) > \frac{U(\bar{x}+U(\underline{x}))}{2} = U(x^\circ) > U(\underline{x})$, and hence the range of the function U is $(0;1)$.
- Moreover, note that if $x \in X'$ such that $x \succ x^\circ$, then $U'(x) \geq U'(\bar{x})$, and thus $x \succsim \bar{x}$.
- Also, it is the case that if $x \in X'$ such that $x^\circ \succ x$, then $U'(\underline{x}) \geq U'(x)$, and thus $\underline{x} \succsim x$.

Proof of the *only if* Direction ($\Rightarrow$) of Proposition 5

Case 4 (continued):

- Firstly, if $x, y \in X'$, then proceed just as in **Case 1**.
- Secondly, if $x \in X'$ and $y = x^\circ$, then $x \succ x^\circ$, if and only if, $x \succsim \overline{x} \succ x^\circ$, if and only if, $U(x) \geq U(\overline{x}) > U(x^\circ)$.
- Thirdly, if $x = x^\circ$ and $y \in X'$, then $x^\circ \succ y$, if and only if, $x^\circ \succ \underline{x} \succsim y$, if and only if, $U(x^\circ) > U(\underline{x}) \geq U(y)$.
- Fourthly, if $x = y = x^\circ$, then proceed just as in **Case 1**.

Proof of the *only if* Direction ($\Rightarrow$) of Proposition 5: Intuitive Recapitulation

- Assume inductively that the representation is possible for sets of size $(n-1)$, and consider some set X of size n as well as some subset X' of size $(n-1)$.

- Produce a representation U' for X', let x° denote the point left out, and investigate where $U(x^\circ)$ should lie.

- It will either be (**Case 1**) equal to some $U'(x')$, or (**Case 2**) to the right of all $U'(x')$, or (**Case 3**) to the left of all $U'(x')$, or (**Case 4**) between two $U'(x')$'s.

- Put $U(x^\circ)$ where it belongs, respectively.

- All the detail in the proof is to show that what results indeed satisfies $x \succ y$, if and only if, $U(x) > U(y)$ for all $x, y \in X$.