ECON322 Game Theory (Half II)

Topic 4: Solution Concepts for Dynamic Games

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Reasoning in Static and Dynamic Games

- **Static games** (typically modelled by the normal-form)
  - Interactive decision situations in which every player independently only makes one choice.
  - When choosing a player does not have any information about his opponents' choices.
  - As no information about the opponents is received during the game, there are no belief changes.

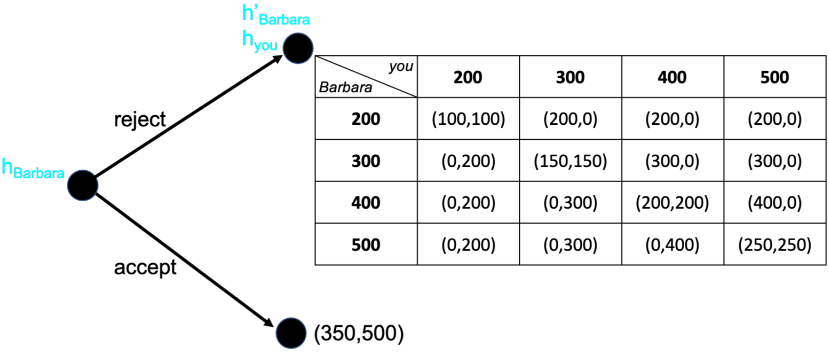
- **Dynamic games** (typically modelled by the extensive-form)
  - Interactive decision situations in which a player independently may have several choices to make.
  - A player may learn about his opponents' previous choices during the game.
  - As a player may observe opponents' choices that he did not expect initially, he may have to revise his beliefs.

Example I: Painting Chris' house

Story:

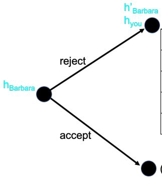
- *Chris* is planning to paint his house tomorrow and needs one person to help.
- *Barbara* and *you* are both interested.
- *Chris* proposes the following procedure: both need to whisper a price from $\{200, 300, 400, 500\}$ in his ear.
- *Chris* will choose the person with the lowest price – in case of a tie, a coin toss will decide on the person to paint with him.
- However, *Barbara* gets a phone call of a colleague to repair his car tomorrow for a price of 350.
- On the phone she needs to decide whether to accept or reject this offer before *Chris* starts his procedure.
- If *Barbara* accepts, then *you* will paint for a price of 500.
- **Question:** What price do you whisper, if *Barbara* rejects?

Example I: Painting Chris' house



Barbara \ you	200	300	400	500
200	(100,100)	(200,0)	(200,0)	(200,0)
300	(0,200)	(150,150)	(300,0)	(300,0)
400	(0,200)	(0,300)	(200,200)	(400,0)
500	(0,200)	(0,300)	(0,400)	(250,250)

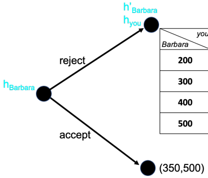
Example I: Painting Chris' house



Barbara \ you	200	300	400	500
200	(100,100)	(200,0)	(200,0)	(200,0)
300	(0,200)	(150,150)	(300,0)	(300,0)
400	(0,200)	(0,300)	(200,200)	(400,0)
500	(0,200)	(0,300)	(0,400)	(250,250)

- A price of 500 can never be optimal for *you*, as it is strictly dominated by the randomized choice $0.5 \cdot 200 + 0.5 \cdot 400$.
- Yet, if *Barbara* believes that you will not choose 500, then she believes not to get more than 300 with reject.
- She would thus be better off by accepting.
- Thus, if *you* believe, that Barbara acts rationally from the beginning onwards and believes that *you* act rationally from h_{you} onwards, then *you* must believe that *Barbara* accepts.

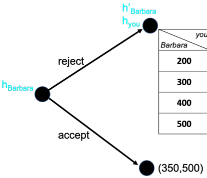
Example I: Painting Chris' house



Barbara \ you	200	300	400	500
200	(100,100)	(200,0)	(200,0)	(200,0)
300	(0,200)	(150,150)	(300,0)	(300,0)
400	(0,200)	(0,300)	(200,200)	(400,0)
500	(0,200)	(0,300)	(0,400)	(250,250)

- However, upon observing reject, *you* believe that *Barbara* has made a mistake and still keep on believing that *Barbara* acts rationally and iteratively thinks in line with rationality from $h'_{Barbara}$ onwards.
- More precisely, *you* still believe that *Barbara* chooses rationally in the remainder of the game; believes you to choose rationally in the remainder of the game; believes you to believe her to choose rationally in the remainder of the game; etc.
- If *you* believe *Barbara* to choose rationally in the remainder of the game, *you* believe she does not choose her irrational choice 500.
- Both 400 and 500 can then no longer be optimal for *you*.
- If *Barbara* believes *you* not to choose 400 and 500, then 200 is uniquely optimal for her.
- Consequently, *you* can only optimally choose 200, too.

Example I: Painting Chris’ house



Barbara \ you	200	300	400	500
200	(100,100)	(200,0)	(200,0)	(200,0)
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- In fact, there also exists another plausible scenario of how to revise your beliefs.
- Suppose that upon observing reject, suppose that *you* try to rationalize – if possible – this choice of hers.
- In fact, *Barbara*’s recject can only be part of a rational strategy, if she chooses 400 at $h'_{Barbara}$ while believing *you* to pick 500.
- Thus, if you strongly believe in rationality, then at h_{you} you must believe that she is implementing her strategy $(reject, 400)$.
- The unique optimal strategy for *you* then is 300.

Example I: Painting Chris' house

Reasonability?

- Both ways of reasoning lead to different optimal choices for *you*.
- They differ in how *you* interpret *Barbara*'s choice of reject.
  - **Scenario 1:** *Barbara* makes a mistake!
  - **Scenario 2:** *Barbara* implements a rational plan!
- Both ways of reasoning seem plausible.
- In fact, there also exists experimental evidence for both ways of reasoning.
- Heterogeneity of agents: different persons reason differently.

Outline

- First of all, attention is restricted to the specific class of dynamic games with perfect information, and the classical solution concept of backward induction is introduced *(using strategies)*.

- Then, the general class of dynamic games with imperfect information is considered.

- The modern solution concepts of backward dominance and forward induction are presented *(using plans)*.

- Both solution concepts correspond to basic and plausible ways of reasoning: common belief in future rationality and common strong belief in rationality.

Agenda

- Backward Induction
- Backward Dominance
- Forward Induction

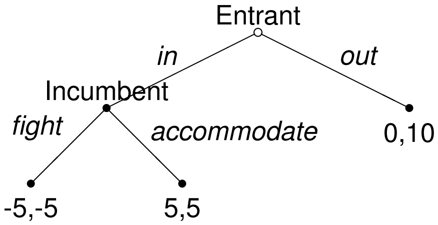
Agenda

- **Backward Induction**
- Backward Dominance
- Forward Induction

Why are New Solution Concepts Needed?

- Why not just form the normal form of a dynamic game and then apply solution concepts for static games?
    - Problem of time.
    - In particular, credibility of a strategy.
- Iterated Strict Dominance – or, equivalently, common belief in rationality – admits non-credible strategies.
- Idea of **sequential rationality**: a strategy specifies optimal behaviour from any point in the game onwards.

Example II: Entry Game



Example II: Entry Game

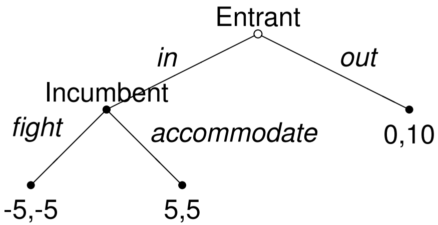
Consider the normal form of the Entry Game:

		Incumbent	
		fight	*accommodate*
Entrant	*in*	$-5, -5$	$5, 5$
	out	$0, 10$	$0, 10$

- Note that $ISD = \{in, out\} \times \{fight, accommodate\}$.
- Therefore, every strategy can be chosen under common belief in rationality.

Example II: Entry Game

- However, the unique best response for the incumbent *after* having observed *in* is *accommodate*.
- The choice *fight* thus violates sequential rationality.
- Hence, *fight* is not a credible strategy for the incumbent against *in*.



Backward Induction

1 At all histories immediately preceding terminal ones, the moving players pick utility-maximizing choices.

2 Taking **1** as given, at all histories immediately preceding only histories from **1** (and possibly terminal ones), the moving players pick utility-maximizing choices.

3 Taking **1** & **2** as given, at all histories immediately preceding only histories from **1**, **2** (and possibly terminal ones), the moving players pick utility-maximizing choices.

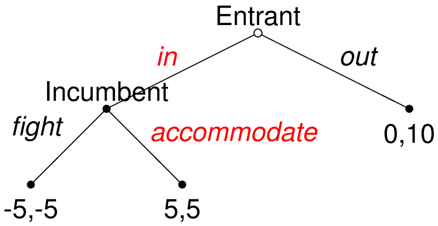
$\vdots$

m Taking **1,2,...,(m-1)** as given, at all histories immediately preceding only histories from **1**, **2**, . . ., **(m-1)** (and possibly terminal ones), the moving players pick utility-maximizing choices.

$\vdots$

END It is proceeded in this fashion up to the root.

Example II: Entry Game



Backward Induction:

$$BI = \{in\} \times \{accommodate\} \subseteq S_{Entrant} \times S_{Incumbent}$$

Remark

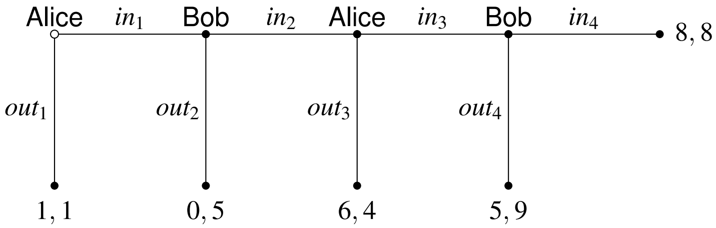
- The requirement of **sequential rationality** is completely captured by Backward Induction for dynamic games with perfect information, since optimal behaviour is specified at every history.

Definition 1

A dynamic game with perfect information is called generic, if for no player $i \in I$ there exist terminal histories $z, z' \in Z$ such that $z \neq z'$ and $U_i(z) = U_i(z')$.

- In every generic dynamic game with perfect information Backward Induction induces a unique strategy profile.

Example III: Centipede



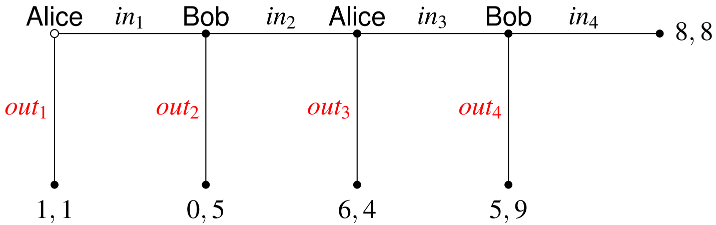
Example III: Centipede

Consider the normal form of the Centipede:

		Bob			
		(in_2, in_4)	(in_2, out_4)	(out_2, in_4)	(out_2, out_4)
Alice	(in_1, in_3)	$8, 8$	$5, 9$	$0, 5$	$0, 5$
	(in_1, out_3)	$6, 4$	$6, 4$	$0, 5$	$0, 5$
	(out_1, in_3)	$1, 1$	$1, 1$	$1, 1$	$1, 1$
	(out_1, out_3)	$1, 1$	$1, 1$	$1, 1$	$1, 1$

- Note that $ISD = \{(in_1, in_3), (in_1, out_3), (out_1, in_3), (out_1, out_3)\} \times \{(in_2, in_4), (in_2, out_4), (out_2, in_4), (out_2, otu_4)\}$.
- Therefore, every strategy can be chosen under common belief in rationality.

Example III: Centipede

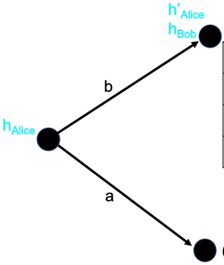


Backward Induction: $BI = \{((out_1, out_3), (out_2, out_4))\} \subseteq S_{Alice} \times S_{Bob}$.

Agenda

- Backward Induction
- **Backward Dominance**
- Forward Induction

Belief in Rationality at Every Information Set is Generally Not Possible!



Alice \ Bob	f	g	e
c	(100,100)	(200,0)	(200,0)
d	(0,200)	(150,150)	(300,0)
e	(0,200)	(0,300)	(200,200)

- At h_{Bob}, *Bob* cannot believe in *Alice*'s rationality, as *Alice*'s unique rational strategy in this game is a.
- What *Bob* can do though is believing in *Alice*'s rationality from h'_{Alice} onwards.
- More generally, a player is said at an information set of his to believe in future rationality, if he believes that all opponents' choose rationally from now onwards.

Formalizing Future Orientation

- Two information sets $h, h' \in \cup_{i \in I} H_i$ are called simultaneous, if there exists a history $x \in X$ such that $x \in h$ and $x \in h'$.
- An information set $h' \in \cup_{i \in I} H_i$ is said to follow some information set $h \in \cup_{i \in I} H_i$, if there exist histories $x \in h$ and $x' \in h'$ such that x' follows x.
- An information set $h' \in \cup_{i \in I} H_i$ is said to weakly follow some information set $h \in \cup_{i \in I} H_i$, if h' follows h or h' and h are simultaneous.

Belief in Future Rationality

- In **dynamic games** players from beliefs at his information sets (i.e. non-terminal histories where he is active.
- As illustrated before it is not always possible to believe an opponent to be rational at all of his information sets.
- But it is always possible to believe an opponent to be rational at all of his future information sets.
- A player i believes at h_i in future rationality, if he believes his opponents to be rational at all h_j weakly following h_i, where $j \neq i$.
- A player is said to believe in future rationality, if at each of his information sets he believes in future rationality.

Full Interactive Thinking: Common Belief in Future Rationality

- A player expresses common belief in future rationality, if
  - he believes in future rationality,
  - he believes at each of his information sets that his opponents believe in future rationality,
  - he believes at each of his information sets that his opponents believe at each of their information sets that their opponents believe in future rationality,
  - etc.
- A plan is then called rational under common belief in future rationality, if it is optimal for the overall conjecture of the player and the player expresses common belief in future rationality.
- This reasoning condition and the respective decision rule can be formalized in **epistemic models** $\mathcal{M}^{\mathcal{E}}$ of **dynamic games**.

Towards a Solution Concept

- A “future orientated” solution concept is now developed based on the idea of belief in future rationality.
- In terms of reasoning the ensueing backward dominance corresponds to common belief in future rationality.

Decision Problems at Information Sets

- Let $\mathcal{E}$ be an extensive form, $i \in I$ some player, and $h_i \in H_i$ some information set of player i.
- The triple $\Gamma_i^0(h_i) = \big(\Phi_i(h_i), \Phi_{-i}(h_i), U_i\big)$ is called full decision problem of i at h_i.
- Any triple $\Gamma_i(h_i) = \big(D_i(h_i), D_{-i}(h_i), U_i|_{D_i(h_i) \times D_{-i}(h_i)}\big)$ is called reduced decision problem of i at h_i, where $D_i(h) \subseteq \Phi_i(h)$ and $D_{-i}(h) \subseteq \Phi_{-i}(h)$.
- A decision problem of i at h_i can be seen as a compressed representation of $\mathcal{E}$ from i's perspective and h_i onwards.

Backward Dominance

Definition 2

Let $\mathcal{E}$ be an extensive form.

- **Step 1:** For every player $i \in I$ and for every information set $h_i \in H_i$, consider the full decision problem $\Gamma_i^0(h_i)$.
  - Eliminate from $\Gamma_i^0(h_i)$ for every player $j \in I$ those plans that are strictly dominated at some full decision problem $\Gamma_j^0(h_j)$ such that h_j weakly follows h_i.
  - A reduced decision problem $\Gamma_i^1(h_i)$ ensues.

Backward Dominance

Definition 2 (continued)

- **Step 2:** For every player $i \in I$ and for every information set $h_i \in H_i$, consider the reduced decision problem $\Gamma_i^1(h_i)$.
  - Eliminate from $\Gamma_i^1(h_i)$ for every player $j \in I$ those plans that are strictly dominated at some reduced decision problem $\Gamma_j^1(h_j)$ such that h_j weakly follows h_i.
  - A reduced decision problem $\Gamma_i^2(h_i)$ ensues.
- **Etc.** until no more plans can be eliminated in this way.

Backward Dominance

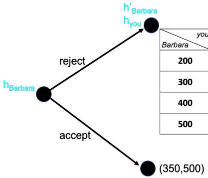
Definition 2 (continued)

- **Output:** Let $k \in I$ be some player that is active at $\emptyset$. The set

$$BD := \times_{i \in I} BD_i$$

is called backward dominance, where for every player $i \in I$ the set BD_i contains all of i's plans in $\cap_{n \geq 0} \Gamma_k^n(\emptyset)$.

Example I: Painting Chris' house



Barbara \ you	200	300	400	500
200	(100,100)	(200,0)	(200,0)	(200,0)
300	(0,200)	(150,150)	(300,0)	(300,0)
400	(0,200)	(0,300)	(200,200)	(400,0)
500	(0,200)	(0,300)	(0,400)	(250,250)

Consider the **full decision problems** for *you* and for *Barbara* at $\emptyset$ and h_1 of the game:

$\Gamma^0_{Barbara}(h_{Barbara})$

	200	300	400	500
(r,200)	100	200	200	200
(r,300)	0	150	300	300
(r,400)	0	0	200	400
(r,500)	0	0	0	250
accept	350	350	350	350

$\Gamma^0_{Barbara}(h'_{Barbara})$

	200	300	400	500
(r,200)	100	200	200	200
(r,300)	0	150	300	300
(r,400)	0	0	200	400
(r,500)	0	0	0	250

$\Gamma^0_{you}(h_{you})$

	(r,200)	(r,300)	(r,400)	(r,500)
200	100	200	200	200
300	0	150	300	300
400	0	0	200	400
500	0	0	0	250

Example I: Painting Chris' house

$\Gamma^0_{Barbara}(h_{Barbara})$

	200	300	400	500
(r,200)	100	200	200	200
(r,300)	0	150	300	300
(r,400)	0	0	200	400
(r,500)	0	0	0	250
accept	350	350	350	350

$\Gamma^0_{Barbara}(h'_{Barbara})$

	200	300	400	500
(r,200)	100	200	200	200
(r,300)	0	150	300	300
(r,400)	0	0	200	400
(r,500)	0	0	0	250

$\Gamma^0_{you}(h_{you})$

	(r,200)	(r,300)	(r,400)	(r,500)
200	100	200	200	200
300	0	150	300	300
400	0	0	200	400
500	0	0	0	250

Step 1

- At $\Gamma^0_{Barbara}(h_{Barbara})$, the plans $(r, 200)$, $(r, 300)$, and $(r, 500)$ are all strictly dominated by *accept* for *Barbara*.
  - Eliminate: $(r, 200)$, $(r, 300)$, and $(r, 500)$ from $\Gamma^0_{Barbara}(h_{Barbara})$.
- At $\Gamma^0_{Barbara}(h'_{Barbara})$ the strictly dominated plans for *Barbara* are $(r, 500)$ by $\frac{1}{2} \cdot (r, 200) + \frac{1}{2} \cdot (r, 400)$ and at $\Gamma^0_{you}(h_{you})$ the strictly dominated plans for *you* are 500 by $\frac{1}{2} \cdot 200 + \frac{1}{2} \cdot 400$.
  - Eliminate: 500 as well as $(r, 500)$ from $\Gamma^0_{Barbara}(h_{Barbara})$, $\Gamma^0_{Barbara}(h'_{Barbara})$, and $\Gamma^0_{you}(h_{you})$.

Example I: Painting Chris' house

$\Gamma^1_{Barbara}(h_{Barbara})$

	200	300	400
(r,400)	0	0	200
accept	350	350	350

$\Gamma^1_{Barbara}(h'_{Barbara})$

	200	300	400
(r,200)	100	200	200
(r,300)	0	150	300
(r,400)	0	0	200

$\Gamma^1_{you}(h_{you})$

	(r,200)	(r,300)	(r,400)
200	100	200	200
300	0	150	300
400	0	0	200

Step 2

- At $\Gamma^1_{Barbara}(h_{Barbara})$, the plan $(r, 400)$ is strictly dominated by *accept* for *Barbara*.
  - Eliminate: $(r, 400)$ from $\Gamma^1_{Barbara}(h_{Barbara})$.
- At $\Gamma^1_{Barbara}(h'_{Barbara})$ the strictly dominated plans for *Barbara* are $(r, 400)$ by $\frac{1}{2} \cdot (r, 200) + \frac{1}{2} \cdot (r, 300)$ and at $\Gamma^1_{you}(h_{you})$ the strictly dominated plans for *you* are 400 by $\frac{1}{2} \cdot 200 + \frac{1}{2} \cdot 300$.
  - Eliminate: 400 as well as $(r, 400)$ from $\Gamma^1_{Barbara}(h_{Barbara})$, $\Gamma^1_{Barbara}(h'_{Barbara})$, and $\Gamma^1_{you}(h_{you})$.

Example I: Painting Chris' house

$\Gamma^2_{Barbara}(h_{Barbara})$

	200	300
accept	350	350

$\Gamma^2_{Barbara}(h'_{Barbara})$

	200	300
(r,200)	100	200
(r,300)	0	150

$\Gamma^2_{you}(h_{you})$

	(r,200)	(r,300)
200	100	200
300	0	150

Step 3

- At $\Gamma^2_{Barbara}(h_{Barbara})$, there exists no strict dominance relationship.
- At $\Gamma^2_{Barbara}(h'_{Barbara})$ the strictly dominated plans for *Barbara* are $(r, 300)$ by $(r, 200)$ and at $\Gamma^2_{you}(h_{you})$ the strictly dominated plans for *you* are 300 by 200.
  - Eliminate: 300 as well as $(r, 300)$ from $\Gamma^2_{Barbara}(h_{Barbara})$, $\Gamma^2_{Barbara}(h'_{Barbara})$, and $\Gamma^2_{you}(h_{you})$.

Example I: Painting Chris' house

$\Gamma^3_{Barbara}(h_{Barbara})$

	200
accept	350

$\Gamma^3_{Barbara}(h'_{Barbara})$

	200
(r,200)	100

$\Gamma^3_{you}(h_{you})$

	(r,200)
200	100

Step 4

- No further plans can be eliminated.
- The procedure **stops**.

Backward Dominance: $BD = BD_{Barbara} \times BD_{you} = \{accept\} \times \{200\} \subseteq \Phi_{Barbara} \times \Phi_{you}$.

Epistemic Characterization of Backward Dominance

Theorem 3

Let $\mathcal{E}$ be an extensive form, $i \in I$ some player, and $\phi_i \in \Phi_i$ some plan of player i. The plan ϕ_i is rational under common belief in future rationaltiy, if and only if, ϕ_i survives backward dominance.

Agenda

- Backward Induction
- Backward Dominance
- **Forward Induction**

Strong Belief in Rationality

- In **dynamic games** it may not always be possible for a player at each of his information sets to believe his opponents are choosing rational plans.

- However, if it is possible for a player i to believe at h_i that all opponents are using rational plans (not avoiding h_i), then i indeed believes at h_i that his opponents are rational.

- In this case i is said to strongly believe in rationality at h_i.

- If it is not possible for i to strongly believe in rationality at h_i, then strong belief in rationality does not restrict i's beliefs at h_i at all.

- Furthermore, a player strongly believes in rationality, if at each of his information sets he strongly believes in rationality.

Full Interactive Thinking: Common Strong Belief in Rationality

- A player expresses common strong belief in rationality, if
  - he strongly believes in rationality,
  - he believes, whenever possible, at each of his information sets that his opponents strongly believe in rationality,
  - he believes, whenever possible, at each of his information sets that his opponents believe at each of their information sets that their opponents strongly believe in rationality,
  - etc.
- A plan is then called rational under common strong belief in rationality, if it is optimal for the overall conjecture of the player and the player expresses common strong belief in rationality.
- This reasoning condition and the respective decision rule can be formalized in **epistemic models** $\mathcal{M}^{\mathcal{E}}$ of **dynamic games**.

Towards a Solution Concept

- A "rationality rigid" solution concept is now developed based on the idea of strong belief in rationality.
- In terms of reasoning the ensueing forward induction corresponds to common strong belief in rationality.

Forward Induction

Definition 4

Let $\mathcal{E}$ be an extensive form.

- **Step 1:** For every player $i \in I$ and for every information set $h_i \in H_i$, consider the full decision problem $\Gamma_i^0(h_i)$.
  - Eliminate from $\Gamma_i^0(h_i)$ for all $j \in I$ the plans being strictly dominated at some full decision problem $\Gamma_j^0(h_j)$ unless this removes all plans not avoiding h_i for some player.
  - A reduced decision problem $\Gamma_i^1(h_i)$ ensues.

Forward Induction

Definition 4 (continued)

- **Step 2:** For every player $i \in I$ and for every information set $h_i \in H_i$, consider the reduced decision problem $\Gamma_i^1(h_i)$.
  - Eliminate from $\Gamma_i^1(h_i)$ for all $j \in I$ the plans being strictly dominated at some reduced decision problem $\Gamma_j^1(h_j)$ unless this removes all plans not avoiding h_i for some player.
  - A reduced decision problem $\Gamma_i^2(h_i)$ ensues.
- **Etc.** until no more plans can be eliminated in this way.

Forward Induction

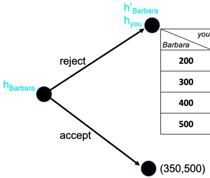
Definition 4 (continued)

- **Output:** Let $k \in I$ be some player that is active at $\emptyset$. The set

$$FI := \times_{i \in I} FI_i$$

is called forward induction, where for every player $i \in I$ the set FI_i contains all of i's plans in $\cap_{n \geq 0} \Gamma_k^n(\emptyset)$.

Example I: Painting Chris' house



Barbara \ you	200	300	400	500
200	(100,100)	(200,0)	(200,0)	(200,0)
300	(0,200)	(150,150)	(300,0)	(300,0)
400	(0,200)	(0,300)	(200,200)	(400,0)
500	(0,200)	(0,300)	(0,400)	(250,250)

Consider the **full decision problems** for *you* and for *Barbara* at $\emptyset$ and h_1 of the game:

$\Gamma^0_{Barbara}(h_{Barbara})$

	200	300	400	500
(r,200)	100	200	200	200
(r,300)	0	150	300	300
(r,400)	0	0	200	400
(r,500)	0	0	0	250
accept	350	350	350	350

$\Gamma^0_{Barbara}(h'_{Barbara})$

	200	300	400	500
(r,200)	100	200	200	200
(r,300)	0	150	300	300
(r,400)	0	0	200	400
(r,500)	0	0	0	250

$\Gamma^0_{you}(h_{you})$

	(r,200)	(r,300)	(r,400)	(r,500)
200	100	200	200	200
300	0	150	300	300
400	0	0	200	400
500	0	0	0	250

Example I: Painting Chris' house

$\Gamma^0_{Barbara}(h_{Barbara})$

	200	300	400	500
(r,200)	100	200	200	200
(r,300)	0	150	300	300
(r,400)	0	0	200	400
(r,500)	0	0	0	250
accept	350	350	350	350

$\Gamma^0_{Barbara}(h'_{Barbara})$

	200	300	400	500
(r,200)	100	200	200	200
(r,300)	0	150	300	300
(r,400)	0	0	200	400
(r,500)	0	0	0	250

$\Gamma^0_{you}(h_{you})$

	(r,200)	(r,300)	(r,400)	(r,500)
200	100	200	200	200
300	0	150	300	300
400	0	0	200	400
500	0	0	0	250

Step 1

- At $\Gamma^0_{Barbara}(h_{Barbara})$, the plans $(r, 200)$, $(r, 300)$, and $(r, 500)$ are all strictly dominated by *accept* for *Barbara*.
    - Eliminate: $(r, 200)$, $(r, 300)$, and $(r, 500)$ from $\Gamma^0_{Barbara}(h_{Barbara})$, $\Gamma^0_{Barbara}(h'_{Barbara})$, and $\Gamma^0_{you}(h_{you})$.
- At $\Gamma^0_{Barbara}(h'_{Barbara})$ the strictly dominated plans for *Barbara* are $(r, 500)$ by $\frac{1}{2} \cdot (r, 200) + \frac{1}{2} \cdot (r, 400)$ and at $\Gamma^0_{you}(h_{you})$ the strictly dominated plans for *you* are 500 by $\frac{1}{2} \cdot 200 + \frac{1}{2} \cdot 400$.
    - Eliminate: 500 as well as $(r, 500)$ from $\Gamma^0_{Barbara}(h_{Barbara})$, $\Gamma^0_{Barbara}(h'_{Barbara})$, and $\Gamma^0_{you}(h_{you})$.

Example I: Painting Chris' house

$\Gamma^1_{Barbara}(h_{Barbara})$

	200	300	400
(r,400)	0	0	200
accept	350	350	350

$\Gamma^1_{Barbara}(h'_{Barbara})$

	200	300	400
(r,400)	0	0	200

$\Gamma^1_{you}(h_{you})$

	(r,400)
200	200
300	300
400	200

Step 2

- At $\Gamma^1_{Barbara}(h_{Barbara})$, the plan $(r, 400)$ is strictly dominated by $accept$ for *Barbara*.
  - Eliminate: $(r, 400)$ from $\Gamma^1_{Barbara}(h_{Barbara})$.
  - However, $(r, 400)$ is not eliminated from neither $\Gamma^1_{Barbara}(h'_{Barbara})$ nor $\Gamma^1_{you}(h_{you})$, as this would remove all plans of *Barbara* at these information sets.
- At $\Gamma^1_{you}(h_{you})$ the strictly dominated plans for *you* are 200 and 400 by 300.
  - Eliminate: 200 as well as 400 from $\Gamma^1_{Barbara}(h_{Barbara})$, $\Gamma^1_{Barbara}(h'_{Barbara})$, and $\Gamma^1_{you}(h_{you})$.

Example I: Painting Chris' house

$\Gamma^2_{Barbara}(h_{Barbara})$

	300
accept	350

$\Gamma^2_{Barbara}(h'_{Barbara})$

	300
(r,400)	0

$\Gamma^2_{you}(h_{you})$

	(r,400)
300	300

Step 3

- No further plans can be eliminated.
- The procedure **stops**.

Forward Induction: $FI = FI_{Barbara} \times FI_{you} = \{accept\} \times \{300\} \subseteq \Phi_{Barbara} \times \Phi_{you}$.

Epistemic Characterization of Forward Induction

Theorem 5

Let $\mathcal{E}$ be an extensive form, $i \in I$ some player, and $\phi_i \in \Phi_i$ some plan of player i. The plan ϕ_i is rational under common strong belief in rationality, if and only if, ϕ_i survives forward induction.

Comparison of the Two Solution Concepts

- The two solution concepts of backward dominance and forward induction have been presented.
- A natural question that arises is how they are related.
  - Is forward induction a refinement of backward dominance?
  - Or is backward dominance a refinement of forward induction?
  - Or are backward dominance and forward induction equivalent?
  - Or is there no logical relationship between backward dominance and forward induciton?

A Deceiving Intuition at First Reflection

- With **BD** a plan is only eliminated at an information set h_i, if it is strictly dominated at a decision problem weakly following h_i.
- With **FI** a plan is only eliminated at an information set h_i, if it is strictly dominated at a decision problem (not necessarily weakly following h_i), unless for the player all plan clear at h_i.
- This might suggest that **FI** eliminates more plans than **BD**.
- However, this intuition is deceiving.

No Logical Relationship between Backward Dominance and Forward Induction

- The reason is the qualification "unless . . . " in **FI**.
- At some step k of **FI**, it may happen that by eliminating from $\Gamma^{k-1}(h_i)$ all plans of some player j, being strictly dominated at some $\Gamma^{k-1}(h_j)$, all plans of j would be removed at $\Gamma^{k-1}(h_i)$.
- In that case, **FI** does not eliminate any plans at h_i, while **BD** could still eliminate some plans at h_i.
- In fact, it can be shown that both solution concepts may yield unique yet distinct plans for the players: thus there exists no logical relationship between **BD** and **FI**.

Special Cases

However, it can be shown that

- in **dynamic games with perfect information** **FI** and **BD** are outcome equivalent.
- in **dynamic games with almost perfect information** the outcomes according to **FI** are included in the outcomes according to **BD**.

Required Background Reading for Topic 4

A. Perea (2012): *Epistemic Game Theory: Reasoning and Choice.* Cambridge University Press.

- Chapter 8 “Belief in the opponents’ future rationality”
- Chapter 9 “Strong belief in the opponents’ rationality”