ECON322 Game Theory (Half II)

Topic 1: Rationality

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ECON322 Half II: Lecturer

- Lecturer: Christian Bach
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- Any questions whatsover:
  - Just send me an email!
  - Or pop by during my office hours:

Thursdays, 3pm-5pm, in room ULMS-CR2

ECON322 Half II: Program

- **Reasoning Foundations for Static Games**
  - Topic 1: Rationality *(T1)*
  - Topic 2: Common Belief in Rationality *(T2)*
- **Dynamic Games**
  - Topic 3: Modelling Dynamic Games *(T3)*
  - Topic 4: Solution Concepts for Dynamic Games *(T4)*
- **Special Event: Friday 19th of November, 9am-11am**
  - Guest Talk by Dr Christian Jaag (CEO, Swiss Economics)
  - Networking Session about a career in economic consulting

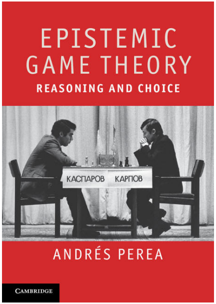
ECON322 Half II: Organization

- "Theory" **Lectures**:
  - weekly 90min sessions on Fridays 9am in room REN-LT7
- "Theory" **Podcasts**:
  - four podcasts accessible via CANVAS
- "Exercise" **Seminars**:
  - Teacher: Tien NGUYEN (hstnguy6@liverpool.ac.uk)
  - Fridays (weeks 9, 10, 11, 12), 1pm-2pm, in room REN-LT7

ECON322 Half II: Assessment

- 2hour exam (closed-book; unseen; on campus) in the January assessment period
- content: **only** Half II is covered in the exam!
- the questions will be similar to the ones discussed in the tutorials
- a mock exam will be provided on Canvas in week 12

ECON322 Half II: Required Reading



- **Chapter 1**: Introduction
- **Chapter 2**: Belief in the opponents' rationality
- **Chapter 3**: Common belief in rationality
- **Chapter 8**: Belief in the opponents' future rationality
- **Chapter 9**: Strong belief in the opponents' rationality

Rationality as a Point of Departure

- In interactive situations ("games") an agent must make a decision, while knowing that the outcome will not only depend on his choice, but also on the choices of other agents.
- Intuitively, in a game an agent makes a choice that he thinks will yield the best outcome to him.
- It is thus crucial what an agent believes his opponents to do.
- In **epistemic game theory** indeed beliefs become the central objects and some intuitive notions can be defined with them.
- A choice is called optimal for an agent, if it yields the best outcome given *his* belief about his opponents' choices.
- A choice is then said to be **rational**, if it is optimal for *some* belief about his opponents' choices.
- **Rationality** typically serves as the primitive, based on which various reasoning concepts are constructed.

Example I: Going to a Party

Story:

- *Alice* and *Bob* are going together to a party tonight.
- *Alice* asks herself what colour she should wear.
- *Alice* prefers *blue* to *green*, *green* to *red*, and *red* to *yellow*.
- However, *Alice* dislikes most to wear the same colour as *Bob*.
- Let *Alice*'s utilities be given as follows:
  - *blue*: 4
  - *green*: 3
  - *red*: 2
  - *yellow*: 1
  - same colour as *Bob*: 0
- **Question:** Which colours can *Alice* **rationally** choose for tonight's party?

Example I: Going to a Party

- *Blue* is optimal for *Alice*, if she believes *Bob* to pick any other colour than *blue*.
- *Green* is optimal for *Alice*, if she believes *Bob* to pick *blue*.
- *Red* is optimal for *Alice*, if she believes that with probability 0.6 *Bob* chooses *blue* and with probability 0.4 *Bob* chooses *green*.
  - Given this belief *Alice* gets 1.6 from *blue* and 1.8 from *green* and 1 from *yellow*
- The colours *blue*, *green*, and *red* are therefore **rational** for *Alice*.

Example I: Going to a Party

- What about the colour *yellow*?

- To see that there is actually no belief such that *yellow* is optimal for *Alice* distinguish two exhaustive cases.

- **Case 1:** Suppose *Alice*'s belief assigns probability of less than 0.5 to *Bob* choosing *blue*. Then, *Alice* expects utility of at least 2 from *blue*, hence *yellow* is not optimal.

- **Case 2:** Suppose *Alice*'s belief assigns probability of at least 0.5 to *Bob* choosing *blue*. Then, *Alice* expects utility of at least 1.5 from *green*, hence *yellow* is not optimal.

- Therefore, *yellow* is **irrational** for *Alice*.

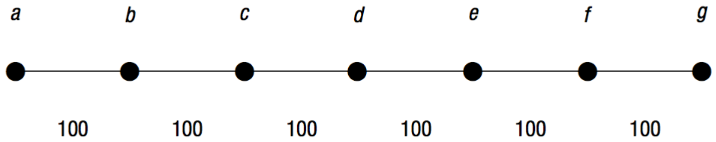
Example II *(For Self-Study)*: Where to Locate My Pub?

Story:

- *Alice* and *Bob* both want to open a new pub on Bold Street.
- Bold Street contains 600 houses, equally spaced.
- One person per house is assumed to visit the closest pub.
- There are seven possible locations for pubs: a, b, c, d, e, f, g.
- If *Alice* and *Bob* choose the same location, then each gets 300 clients.
- **Question:** Which location should *Alice* choose to maximize her benefits, i.e. the number of clients?
- It depends on the location *Alice* believes *Bob* to choose!

Example II *(For Self-Study)*: Where to Locate My Pub?

Bold Street:



Example II *(For Self-Study)*: Where to Locate My Pub?

- Suppose *Alice* believes *Bob* to choose *a*.
  Then, *b* is optimal for *Alice*.
- Suppose *Alice* believes *Bob* to choose *b*.
  Then, *c* is optimal for *Alice*.
- Suppose *Alice* believes *Bob* to choose *c*.
  Then, *d* is optimal for *Alice*.
- Suppose *Alice* believes *Bob* to choose *d*.
  Then, *d* is optimal for *Alice*.
- Suppose *Alice* believes *Bob* to choose *e*.
  Then, *d* is optimal for *Alice*.
- Suppose *Alice* believes *Bob* to choose *f*.
  Then, *e* is optimal for *Alice*.
- Suppose *Alice* believes *Bob* to choose *g*.
  Then, *f* is optimal for *Alice*.

Example II *(For Self-Study)*: Where to Locate My Pub?

- The choices b, c, d, e, and f are therefore rational for *Alice*.
- What about the choices a and g?
- Whatever *Alice* believes *Bob* to choose, b is better than a, and f is better than g.
- The choices a and g are therefore irrational for *Alice*.

Agenda

- Rational Choice
- Strict Dominance
- Characterization of Rationality

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- **Rational Choice**
- Strict Dominance
- Characterization of Rationality

Games

Definition 1

The tuple $\Gamma_N = \big(I, (C_i)_{i \in I}, (U_i)_{i \in I}\big)$ is called normal form, where

- I denotes the finite set of *players*,
- C_i denotes the finite set of *choices* of player i,
- $U_i : \times_{j \in I} C_j \to \mathbb{R}$ denotes the *utility function* of player i.

Belief About the Opponents' Choices

Definition 2

Let Γ_N be a normal form, and i be a player. A *belief for player i about the opponents' choices* – also called conjecture – is a probability distribution

$$\beta_i : C_{-i} \to [0, 1]$$

over the set of opponents' choice combinations $C_{-i} := \times_{j \in I \setminus \{i\}} C_j$.

Note that the set of all probability distributions on some set X is often denoted by $\Delta(X) := \{p \in [0, 1]^X : \sum_{x \in X} p(x) = 1\}$.

Expected Utility

Definition 3

Let Γ_N be a normal form, and i be a player. Suppose that player i entertains conjecture β_i and chooses c_i. The *expected utility for player i* is

$$u_i(c_i, \beta_i) := \sum_{c_{-i} \in C_{-i}} \beta_i(c_{-i}) \cdot U_i(c_i, c_{-i})$$

where $(c_i, c_{-i}) := (c_1, \ldots, c_n) \in \times_{j \in I} C_j$.

Optimality

Definition 4

Let Γ_N be a normal form, and i be a player. Suppose that player i entertains conjecture β_i. A choice c_i for player i is *optimal given conjecture β_i*, if

$$u_i(c_i, \beta_i) \geq u_i(c_i', \beta_i)$$

holds for all choices $c_i' \in C_i$ of player i.

Rationality

Definition 5

Let Γ_N be a normal form, and i be a player. A choice c_i for player i is *rational*, if there exists a conjecture β_i for player i such that c_i is optimal given β_i.

Illustration

Bob

Alice		L	R
	U	$10, 5$	$0, 3$
	M	$0, 2$	$10, 2$
	D	$7, -3$	$7, 1$

All three choices for *Alice* are rational.

- *U* is optimal for *Alice*, if she believes *Bob* to choose *L*.
- *M* is optimal for *Alice*, if she believes *Bob* to choose *R*.
- *D* is optimal for *Alice*, if she believes with probability 0.5 *Bob* to choose *L* and with probability 0.5 *Bob* to choose *R*.
  (Note that if a choice is not optimal for any probability-1 belief, then it can still be optimal for some belief with supp > 1)

Agenda

- Rational Choice
- **Strict Dominance**
- Characterization of Rationality

Randomizing

Definition 6

Let Γ_N be a normal form, and i be a player. A *mixed choice* for player i is a probability distribution

$$\sigma_i : C_i \to [0, 1]$$

over the set C_i of player i's choices.

Remark:

- It seems unnatural that people randomize when taking serious decisions.
- We assume players to make definite decisions.
- However, mixed strategies are used as technical tools for identifying the rational (pure) choices in games.

Utility with Randomizing

Definition 7

Let Γ_N be a normal form, and i be a player. Suppose that player i chooses σ_i, and that his opponents choose according to c_{-i}. The *randomizing-utility for player i* is

$$V_i(\sigma_i, c_{-i}) = \sum_{c_i \in C_i} \sigma_i(c_i) \cdot U_i(c_i, c_{-i})$$

where $(c_i, c_{-i}) = (c_1, \ldots, c_n) \in \times_{j \in I} C_j$.

Expected Utility with Randomizing

Definition 8

Let Γ_N be a normal form, and i be a player. Suppose that player i entertains conejcture β_i and chooses σ_i. The *expected randomizing-utility for player i* is

$$v_i(\sigma_i, \beta_i) = \sum_{c_{-i} \in C_{-i}} \beta_i(c_{-i}) \cdot V_i(\sigma_i, c_{-i})$$

$$= \sum_{c_{-i} \in C_{-i}} \beta_i(c_{-i}) \cdot \Big(\sum_{c_i \in C_i} \sigma_i(c_i) \cdot U_i(c_i, c_{-i}) \Big)$$

where $(c_i, c_{-i}) = (c_1, \dots, c_n) \in \times_{j \in I} C_j$.

Strict Dominance: The Pure Case

Definition 9

Let Γ_N be a normal form, and i be a player. A choice c_i for player i is *strictly pure-dominated*, if there exists some choice $c_i' \in C_i$ of player i such that

$$U_i(c_i, c_{-i}) < U_i(c_i', c_{-i})$$

holds for every opponents' choice combination $c_{-i} \in C_{-i}$.

Strict Dominance: The Randomized Case

Definition 10

Let Γ_N be a normal form, and i be a player. A choice c_i for player i is *strictly randomized-dominated*, if there exists some mixed choice $\sigma_i \in \Delta(C_i)$ of player i such that

$$U_i(c_i, c_{-i}) < V_i(\sigma_i, c_{-i})$$

holds for every opponents' choice combination $c_{-i} \in C_{-i}$.

Strict Dominance

Definition 11

Let Γ_N be a normal form, and i be a player. A choice c_i for player i is *strictly dominated*, if c_i is either strictly pure-dominated or strictly randomized-dominated.

Example I: Going to a Party

Story:

- *Alice* and *Bob* are going together to a party tonight.
- *Alice* asks herself what colour she should wear.
- *Alice* prefers *blue* to *green*, *green* to *red*, and *red* to *yellow*.
- However, *Alice* dislikes most to wear the same colour as *Bob*.
- Let *Alice*'s utilities be given as follows:
  - *blue*: 4
  - *green*: 3
  - *red*: 2
  - *yellow*: 1
  - same colour as *Bob*: 0
- **Question:** Which colour choices obtain with strict dominance for *Alice*?

Example I: Going to a Party

- Neither *blue*, nor *green*, nor *red* are strictly dominated for *Alice*:
  - $U_{Alice}(blue, green) \geq U_{Alice}(c_{Alice}, green)$ for all $c_{Alice} \in \{blue, green, red, yellow\}$,
  - $U_{Alice}(green, blue) \geq U_{Alice}(c_{Alice}, blue)$ for all $c_{Alice} \in \{blue, green, red, yellow\}$,
  - $U_{Alice}(red, blue) > U_{Alice}(blue, blue)$, $U_{Alice}(red, green) > U_{Alice}(green, green)$, and $U_{Alice}(red, yellow) > U_{Alice}(yellow, yellow)$, hence there cannot be a pure choice for *Alice* better than *red* against each of *Bob*'s choices.
- *yellow* is strictly dominated by $0.5 \cdot blue + 0.5 \cdot green$ for *Alice*, as

$$U_{Alice}(yellow, c_{Bob}) < V_{Alice}(0.5 \cdot blue + 0.5 \cdot green, c_{Bob})$$

  for all $c_{Bob} \in \{blue, green, red, yellow\}$.
- Hence, $SD_{Alice} = \{blue, green, red\}$.

Example II *(For Self-Study)*: Where to Locate My Pub?

Story:

- *Alice* and *Bob* both want to open a new pub on Bold Street.
- Bold Street contains 600 houses, equally spaced.
- One person per house is assumed to visit the closest pub.
- There are seven possible locations for pubs: a, b, c, d, e, f, g.
- If *Alice* and *Bob* choose the same location, then each gets 300 clients.
- **Question:** Which location choices obtain with strict dominance?

Example II *(For Self-Study)*: Where to Locate My Pub?

- Neither *b*, nor *c*, nor *d*, *e*, nor *f* are strictly dominated for *Alice*:

  - $U_{Alice}(b,a) \geq U_{Alice}(c_{Alice},a)$ for all $c_{Alice} \in \{a,b,c,d,e,f,g\}$,
  - $U_{Alice}(c,b) \geq U_{Alice}(c_{Alice},b)$ for all $c_{Alice} \in \{a,b,c,d,e,f,g\}$,
  - $U_{Alice}(d,d) \geq U_{Alice}(c_{Alice},d)$ for all $c_{Alice} \in \{a,b,c,d,e,f,g\}$,
  - $U_{Alice}(e,f) \geq U_{Alice}(c_{Alice},f)$ for all $c_{Alice} \in \{a,b,c,d,e,f,g\}$,
  - $U_{Alice}(f,g) \geq U_{Alice}(c_{Alice},g)$ for all $c_{Alice} \in \{a,b,c,d,e,f,g\}$.

- a is strictly dominated by b for *Alice*, as $U_{Alice}(a,c_{Bob}) < U_{Alice}(b,c_{Bob})$ for all $c_{Bob} \in \{a,b,c,d,e,f,g\}$.

- g is strictly dominated by f for *Alice*, as $U_{Alice}(g,c_{Bob}) < U_{Alice}(f,c_{Bob})$ for all $c_{Bob} \in \{a,b,c,d,e,f,g\}$.

- Hence, $SD_{Alice} = \{b,c,d,e,f\}$, and analogously, it can be shown that, $SD_{Bob} = \{b,c,d,e,f\}$.

Agenda

- Rational Choice
- Strict Dominance
- **Characterization of Rationality**

Characterization of Rationality

Pearce's Lemma:
The rational choices are exactly those choices that are not strictly dominated.

Application

Four ways to rationality:

1. Identify all rational choices:
   find a conjecture such that the respective choice is optimal.
2. Identify all irrational choices:
   show that the respective choice is not optimal for any conjecture.
3. Identify all choices that are not strictly dominated:
   find an opponents' choice-combination such that there is no choice that is better than the respective choice.
4. Identify all choices that are strictly dominated:
   show that the respective choice fares worse than some mixed choice (or some other pure choice) for all opponents' choice-combinations.

Note:

- For rational choices it is often easier to find a supporting belief.
- For irrational choices it is often easier to show strict dominance.

Required Background Reading for Topic 1

A. Perea (2012): *Epistemic Game Theory: Reasoning and Choice.* Cambridge University Press.

- Chapter 1 “Introduction”
- Chapter 2 “Belief in the opponents’ rationality”.